Autoequivalences of derived categories of elliptic surfaces with non-zero Kodaira dimension

Hokuto Uehara

Abstract

We study the group of autoequivalences of the derived categories of coherent sheaves on smooth projective elliptic surfaces with non-zero Kodaira dimension. We find a description of it when each reducible fiber is a cycle of (-2)-curves and non-multiple.

Contents

1 Introduction **2**
- 1.1 Motivations and results 2
- 1.2 Outline of the proof of Theorem 1.3 5
- 1.3 Notation and conventions 6
- 1.4 Acknowledgements 6

2 Preliminaries **6**
- 2.1 General results for Fourier–Mukai transforms 6
- 2.2 The Euler form on surfaces 9
- 2.3 Twist functors 9
- 2.4 Autoequivalences of elliptic curves 11
- 2.5 Automorphisms of elliptic surfaces 12
- 2.6 Fourier–Mukai transforms on elliptic surfaces 12

3 Autoequivalences of elliptic surfaces with non-zero Kodaira dimension **15**
- 3.1 Notation and the setting 15
- 3.2 Autoequivalences associated with reducible fibers 17
- 3.3 Autoequivalences associated with irreducible fibers: The classification 19
- 3.4 Autoequivalences associated with irreducible fibers: The structure of the group 21
- 3.5 Kernels of ι_Z and ι_U 24

4 Autoequivalences associated with singular fibers of type I_n for $n > 1$ **25**
4.1 Reduction of the proof of Theorem 4.1 25
4.2 The cohomology sheaves of spherical objects 27

5 The proof of Proposition 4.4 **33**
5.1 Auxiliary results 33
5.2 The proof of Proposition 4.4 38

6 The proof of Proposition 4.2 **38**
6.1 Facts needed for the proof of Proposition 4.2 38
6.2 Case (i) 40
6.3 Case (ii) 41
6.4 Case (iii) 41

7 Example **43**

1 Introduction

1.1 Motivations and results

Let X be a smooth projective variety over $\mathbb{C}$ and $D(X)$ the bounded derived category of coherent sheaves on X. If X and Y are smooth projective varieties with equivalent derived categories, then we call X and Y *Fourier–Mukai partners*. We define the set of isomorphism classes of Fourier–Mukai partner of X as

$$\mathrm{FM}(X) := \{Y \text{ smooth projective varieties } \mid D(X) \cong D(Y)\}/\cong .$$

It is an interesting problem to determines the set $\mathrm{FM}(X)$ for a given X. There are several known results in this direction. For example, Bondal and Orlov show that if K_X or $-K_X$ is ample, then X can be entirely reconstructed from $D(X)$, namely $\mathrm{FM}(X) = \{X\}$ [BO95]. To the contrary, there are examples of non-isomorphic varieties X and Y having equivalent derived categories. For example, in dimension 2, if $\mathrm{FM}(X) \neq \{X\}$, then X is a K3 surface, an abelian surface or a relatively minimal elliptic surface with non-zero Kodaira dimension ([BM01], [Ka02]). In dimension 3, some results are shown by Toda [To03]. Moreover, Orlov gives a complete answer to this problem for abelian varieties in [Or02].

It is also natural to study the isomorphism classes of autoequivalences of $D(X)$. The group consisting of all exact $\mathbb{C}$-linear autoequivalences of $D(X)$ up to isomorphism is denoted by

$$\operatorname{Auteq} D(X).$$

We note that $\operatorname{Auteq} D(X)$ always contains the group

$$A(X) := \operatorname{Pic} X \rtimes \operatorname{Aut} X \times \mathbb{Z}[1],$$

generated by *standard autoequivalences*, namely the functors of tensoring with invertible sheaves, pull backs along automorphisms, and the shift functor [1].

When K_X or $-K_X$ is ample, Bondal and Orlov show that $\operatorname{Auteq} D(X) \cong A(X)$.

When X is an abelian variety, Orlov determines the structure of $\operatorname{Auteq} D(X)$ ([Or02]). As a special case, when X is an elliptic curve, the autoequivalence group is described as

$$1 \to \hat{X} \rtimes \operatorname{Aut} X \times \mathbb{Z}[2] \to \operatorname{Auteq} D(X) \xrightarrow{\theta} \mathrm{SL}(2, \mathbb{Z}) \to 1.$$

Here θ is given by the action of $\operatorname{Auteq} D(X)$ on the even integral cohomology group $H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$, which is isomorphic to $\mathbb{Z}^2$. In this case, the group $\operatorname{Auteq} D(X)$ contains the Fourier–Mukai transform $\Phi^{\mathcal{U}}_{J_X(a,b)\to X}$ determined by a universal sheaf $\mathcal{U}$ of the fine moduli space $J_X(a, b)$ of stable vector bundles of the rank a and the degree b with $(a, b) = 1$. By the work of Atiyah ([At57, Theorem 7]), $J_X(a, b)$ is isomorphic to X, and hence $\Phi^{\mathcal{U}}_{J_X(a,b)\to X}$ can be regarded as an autoequivalence of $D(X)$. One can check that $\Phi^{\mathcal{U}}_{J_X(a,b)\to X}$ does not belong to $A(X)$.

For the minimal resolution X of A_n-singularities on a surface, Ishii and the author determine the structure of $\operatorname{Auteq} D(X)$ ([IU05]). It is generated by the group $A(X)$ and twist functors of the form $T_{\mathcal{O}_G(a)}$ ([ST01]) associated with the line bundle $\mathcal{O}_G(a)$ on a (-2)-curve $G(\cong \mathbb{P}^1)$ on X. Again, $T_{\mathcal{O}_G(a)}$ does not belong to $A(X)$.

The case of smooth projective elliptic surfaces $\pi\colon S \to C$ with non-zero Kodaira dimension is a mixture of the last two cases. If S has a reducible fiber, each component of it is a (-2)-curve. Hence $\operatorname{Auteq} D(S)$ contains twist functors as in the case [IU05]. On the other hand, let us consider the fine moduli space $J_S(a, b)$ of pure 1-dimensional stable sheaves on S, the general point of which represents a rank a, degree b stable vector bundle supported on a smooth fiber of π. It often occurs that there is an isomorphism $S \cong J_S(a, b)$, and then the Fourier–Mukai transform $\Phi^{\mathcal{U}}_{J_S(a,b)\to S}$ determined by a universal sheaf $\mathcal{U}$ on $J_S(a, b) \times S$ gives a non-standard autoequivalence.

For an object E of $D(S)$, we define the fiber degree of E

$$d(E) = c_1(E) \cdot F,$$

where F is a general fiber of π. Let us denote by λ_S the highest common factor of the fiber degrees of objects of $D(S)$. It is shown in [Br98] that if $a\lambda_S$ and b are coprime, the above mentioned fine moduli space $J_S(a, b)$ exists. We denote $J_S(b) := J_S(1, b)$.

We set

$$B := \left\langle T_{\mathcal{O}_G(a)} \mid G \text{ is a } (-2)\text{-curve } \right\rangle$$

and denote the congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ by

$$\Gamma_0(m) := \left\{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; d \in m\mathbb{Z} \right\}$$

for $m \in \mathbb{Z}$.

Conjecture 1.1. *Let S be a smooth projective elliptic surface S with $\kappa(S) \neq 0$. Then we have a short exact sequence*

$$1 \to \langle B, \otimes\mathcal{O}_S(D) \mid D \cdot F = 0,\ F \textit{ is a fiber}\rangle \rtimes \operatorname{Aut} S \times \mathbb{Z}[2] \to \operatorname{Auteq} D(S) \xrightarrow{\Theta} \left\{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \Gamma_0(\lambda_S) \;\middle|\; J_S(b) \cong S \right\} \to 1.$$

Here Θ is induced by the action of $\operatorname{Auteq} D(S)$ on the even degree part $H^0(F,\mathbb{Z}) \oplus H^2(F,\mathbb{Z}) \cong \mathbb{Z}^2$ of integral cohomology groups of on a smooth fiber F.

Remark 1.2. (i) As Conjecture 1.1 implicitly implies, we can actually see in the proof of Theorem 3.11 that every element of $\operatorname{Auteq} D(S)$ induces an autoequivalence of a smooth fiber F. If $\kappa(S) = 0$, this is false. See in Example 2.8 (iii).

(ii) The quotient group $\Gamma_0(\lambda_S)/\operatorname{Im}\Theta$ is naturally identified with the set of Fourier–Mukai partners $\mathrm{FM}(S)$ of S. See Remark 3.12.

(iii) If π has a section, we know that $\operatorname{Im}\Theta \cong \mathrm{SL}(2,\mathbb{Z})$. See Remark 3.12.

(iv) By [IU05, Proposition 4.18], we have

$$B \cap \langle \otimes\mathcal{O}_S(D) \mid D \cdot F = 0,\ F \text{ is a fiber}\rangle = \langle \otimes\mathcal{O}_S(G) \mid\ G \text{ is a } (-2)\text{-curve}\rangle .$$

The following is the main result in this article.

Theorem 1.3 (=Theorem 4.1)**.** *Suppose that each reducible fiber on the elliptic surface S is non-multiple, and forms a cycle of (-2)-curves, i.e. of type I_n for some $n > 1$. Then Conjecture 1.1 is true.*

In the case that π has only irreducible fibers, and π has a section, Conjecture 1.1 is essentially shown in [LST13].[1]

[1] They only consider autoequivalences Φ with $\Phi \in \operatorname{Ker}\delta$. See the definition of δ in Remark 3.10. On the other hand, their result is valid without any restrictions on the base space of the fibration.

1.2 Outline of the proof of Theorem 1.3

Let S be a projective elliptic surface with $\kappa(S) \neq 0$, and Z be the union of all reducible fibers of the elliptic fibration $\pi\colon S \to C$, and U be the complement of Z in S. We introduce a group homomorphism

$$\iota_U\colon \operatorname{Auteq} D(S) \to \operatorname{Auteq} D(U)$$

in Proposition 3.9 and denote $\operatorname{Im} \iota_U$ by $\operatorname{Auteq}^\dagger D(U)$. We classify all elements of $\operatorname{Auteq}^\dagger D(U)$ by Proposition 3.6 and determine the structure of $\operatorname{Auteq}^\dagger D(U)$ in Theorem 3.11.

Assume furthermore that all reducible fibers of π are of type I_n $(n > 1)$ as in Theorem 1.3. Then we can show that

$$B = \operatorname{Ker} \iota_U \tag{1}$$

by using

Proposition 1.4 (=Proposition 4.2)**.** *Take a connected component Z_0 of Z, that means a fiber of π. Let us consider the irreducible decomposition $Z_0 = C_1 \cup \cdots \cup C_n$, where each C_i is a (-2)-curve. Suppose that we are given an autoequivalence Φ of $D_{Z_0}(S)$ preserving the cohomology class $[\mathcal{O}_x] \in H^4(S, \mathbb{Q})$ for some point $x \in Z_0$. Then, there are integers a, b $(1 \le b \le n)$ and i, and there is an autoequivalence*

$$\Psi \in \left\langle T_{\mathcal{O}_{G(a)}} \mid G \textit{ is a } (-2)\textit{-curves contained in } Z_0 \right\rangle$$

such that

$$\Psi \circ \Phi(\mathcal{O}_{C_1}) \cong \mathcal{O}_{C_b}(a)[i]$$

and

$$\Psi \circ \Phi(\mathcal{O}_{C_1}(-1)) \cong \mathcal{O}_{C_b}(a-1)[i].$$

In particular, for any point $x \in C_1$, we can find a point $y \in C_b$ with $\Psi \circ \Phi(\mathcal{O}_x) \cong \mathcal{O}_y[i]$.

Proposition 1.4 is proved in §6 using techniques developed in [IU05]. Then we can deduce Theorem 1.3 from the equation (1) and the description of $\operatorname{Auteq}^\dagger D(U)$ obtained in Theorem 3.11.

The construction of this article is as follows. In §2 we show several preliminary results and give definitions needed afterwards. In §3, we study the structure of $\operatorname{Auteq}^\dagger D(U)$ intensively. In §4, we reduce the proof of Theorem 1.3 to showing the equation (1), and furthermore we reduce the proof of (1) to showing Proposition 1.4. In §4, we also show several lemmas used in §5 and §6. In §5, we show Proposition 4.4, which is the first step in the proof of Proposition 1.4. In §6, we prove Proposition 1.4. Finally in §7, we treat an example of elliptic surfaces satisfying the assumption in Theorem 1.3. In the example, we can determine the set $\mathrm{FM}(S)$, and also know when $J_S(b) \cong S$ holds. This information gives us a better description of $\operatorname{Im} \Theta$ (see Conjecture 1.1) in the example.

1.3 Notation and conventions

All varieties will be defined over $\mathbb{C}$. A *point* on a variety will always mean a closed point. By an *elliptic surface*, we will always mean a smooth surface S together with a smooth curve C and a relatively minimal projective morphism $\pi\colon S \to C$ whose general fiber is an elliptic curve. Here a *relatively minimal morphism* means a morphism whose fibers contains no (-1)-curves.

For two elliptic surfaces $\pi\colon S \to C$ and $\pi'\colon S' \to C$, an isomorphism $\varphi\colon S \to S'$ satisfying $\pi = \pi' \circ \varphi$ is called an *isomorphism over* C.

For an elliptic curve E and some positive integer m, we denote the set of points of order m by ${}_mE$. Furthermore, we denote the dual elliptic curve by $\hat{E} := \operatorname{Pic}^0 E$.

$D(X)$ denotes the bounded derived category of coherent sheaves on an algebraic variety X. For a closed subset Z of X, we denote the full subcategory of $D(X)$ consisting of objects supported on Z by $D_Z(X)$. Here, the support of an object of $D(X)$ is, by definition, the union of the set-theoretic supports of its cohomology sheaves.

An object α in $D(X)$ is said to be *simple* (respectively *rigid*) if

$$\operatorname{Hom}_{D(X)}(\alpha, \alpha) \cong \mathbb{C} \text{ (respectively } \operatorname{Hom}^1_{D(X)}(\alpha, \alpha) \cong 0).$$

Given a closed embedding of schemes $i\colon Z \hookrightarrow X$, we often denote the derived pull back $\mathbb{L}i^*\alpha$ simply by $\alpha|_Z$.

$\operatorname{Auteq} \mathcal{T}$ denotes the group of isomorphism classes of $\mathbb{C}$-linear exact autoequivalences of a $\mathbb{C}$-linear triangulated category $\mathcal{T}$.

1.4 Acknowledgements

The author is supported by the Grants-in-Aid for Scientific Research (No.23340011). This paper was written during his staying at the Max-Planck Institute for Mathematics, in the period from April to September 2014. The author appreciates the hospitality. He also thanks Yukinobu Toda for pointing out some mistakes in the first draft.

2 Preliminaries

2.1 General results for Fourier–Mukai transforms

Let X and Y be smooth projective varieties. We call Y *a Fourier–Mukai partner of* X if $D(X)$ is $\mathbb{C}$-linear triangulated equivalent to $D(Y)$. We denote by $\operatorname{FM}(X)$ the set of isomorphism classes of Fourier–Mukai partners of X.

For an object $\mathcal{P} \in D(X \times Y)$, we define an exact functor $\Phi^{\mathcal{P}}$, called an *integral functor*, to be

$$\Phi^{\mathcal{P}} := \mathbb{R}p_{Y*}(\mathcal{P} \overset{\mathbb{L}}{\otimes} p_X^*(-))\colon D(X) \to D(Y),$$

where we denote the projections by $p_X\colon X\times Y\to X$ and $p_Y\colon X\times Y\to Y$. We also sometimes write $\Phi^{\mathcal{P}}$ as $\Phi^{\mathcal{P}}_{X\to Y}$ to emphasize that it is a functor from $D(X)$ to $D(Y)$.

Next suppose that X and Y are not necessarily projective. Then, in general, $\mathbb{R}p_{Y*}$ is not well-defined as a functor $D(X\times Y)\to D(Y)$ since p_Y is not projective. Instead, suppose that there are projective morphisms $X\to C$ and $Y\to C$ over a smooth variety C, and let $\mathcal{P}$ be a perfect complex in $D(X\times_C Y)$. Then we can also define the integral functor in this case, by replacing the projections p_Y and p_X with $p_Y\colon X\times_C Y\to Y$ and $p_X\colon X\times_C Y\to X$ respectively (note that we use the same notation for both kinds of projections). If we want to emphasize that we are in this situation, Φ is called a *relative integral functor over C*. Later, we use relative integral transforms in the case of elliptic surfaces over a non-projective base C.

By the result of Orlov ([Or97]), for smooth projective varieties X and Y, and for a fully faithful functor $\Phi\colon D(X)\to D(Y)$, there is an object $\mathcal{P}\in D(X\times Y)$, unique up to isomorphism, such that

$$\Phi\cong\Phi^{\mathcal{P}}.$$

If an integral functor (over C) is an equivalence, it is called a *Fourier–Mukai transform (over C)*.

The left adjoint to an integral functor $\Phi^{\mathcal{P}}$ over C is given by the integral functor $\Phi^{\mathcal{Q}}$ over C where

$$\mathcal{Q}:=\mathbb{R}\mathcal{H}om_{X\times_C Y}(\mathcal{P},\mathcal{O}_{X\times_C Y})\overset{\mathbb{L}}{\otimes}p_X^*\omega_{X/C}[\dim X-\dim C]$$

(see the proof of [Hu06, Proposition 5.9]). In particular, if $\Phi^{\mathcal{P}}$ is an equivalence, its quasi-inverse is given by $\Phi^{\mathcal{Q}}$.

We can also see that the composition of integral functors over C is again an integral functor over C (cf. [Hu06, Proposition 5.10]).

Lemma 2.1. *Let $\Phi\colon D(X)\to D(Y)$ be a Fourier–Mukai transform over a smooth variety C between smooth varieties X,Y, projective over C. Then the set of points $x\in X$ for which the object $\Phi(\mathcal{O}_x)$ is a sheaf forms an (possibly empty) open subset of X.*

Proof. This is a special case of [BM01, Proposition 2.4]. See also the proof of [BM01, Lemma 2.5]. □

The following is well-known.

Lemma 2.2. *Let $\Phi\colon D(X)\to D(Y)$ be a Fourier–Mukai transform over a smooth variety C between smooth varieties X,Y, projective over C. Assume that it satisfies that $\Phi(\mathcal{O}_x)$ is a shift of a sheaf supported on a finite subset of Y for all points $x\in X$. Then we have*

$$\Phi\cong\phi_*\circ((-)\otimes\mathcal{L})[n]$$

for a line bundle $\mathcal{L}$ on X, an isomorphism $\phi\colon X \to Y$ over C and some integer n.

Proof. By the assumptions, $\Phi(\mathcal{O}_x)$ satisfies the condition of [Hu06, Lemma 4.5]. Hence, $\Phi(\mathcal{O}_x) \cong \mathcal{O}_y[n]$ for some $y \in Y$ and $n \in \mathbb{Z}$. Note that the integer n does not depend on the choice of a point x by Lemma 2.1. Then apply [BM98, §3.3] (or [Hu06, Corollary 5.23]) to get the conclusion. $\square$

The following Lemma is useful.

Lemma 2.3. *Let X be a smooth variety. For an object $E \in D(X)$ with a compact support, $\mathbb{R}\mathrm{Hom}_{D(X)}(E, \mathcal{O}_x) \neq 0$ if and only if a point x is contained in $\operatorname{Supp} E$. Moreover, these conditions are also equivalent to $\mathbb{R}\mathrm{Hom}_{D(X)}(\mathcal{O}_x, E) \neq 0$.*

Proof. The first statement is just [BM02, Lemma 5.3]. The second follows from the Grothendieck–Serre duality and the first. $\square$

Lemma 2.4. *Let X and Y be smooth projective varieties together with closed subsets $Z \subset X$ and $W \subset Y$. Suppose that a Fourier–Mukai transform $\Phi = \Phi_{X \to Y}$ and its quasi-inverse Ψ satisfy that*

$$\operatorname{Supp} \Phi(\mathcal{O}_x) \subset W \text{ and } \operatorname{Supp} \Psi(\mathcal{O}_y) \subset Z.$$

for any point $x \in Z, y \in W$ Then Φ restricts to a Fourier–Mukai transform from $D_Z(X)$ to $D_W(Y)$.

Proof. We repeatedly use Lemma 2.3. Take a point $x \in Z$ and $y \in Y \backslash W$. Then

$$\mathbb{R}\mathrm{Hom}_{D(X)}(\Psi(\mathcal{O}_y), \mathcal{O}_x) = \mathbb{R}\mathrm{Hom}_{D(Y)}(\mathcal{O}_y, \Phi(\mathcal{O}_x))$$

vanishes by the assumption. This means that $\operatorname{Supp} \Psi(\mathcal{O}_y) \cap Z = \emptyset$, and thus for an object $E \in D_Z(X)$, we have

$$\mathbb{R}\mathrm{Hom}_{D(Y)}(\Phi(E), \mathcal{O}_y) = \mathbb{R}\mathrm{Hom}_{D(X)}(E, \Psi(\mathcal{O}_y)) = 0,$$

which implies that $\Phi(E) \in D_W(Y)$. Here, the last equality follows from [Hu06, Lemma 3.9]. In a similar way, we can prove that $\Psi(F) \in D_Z(S)$ for any objects $F \in D_W(Y)$. Therefore, we obtain the conclusion. $\square$

The following is also well-known.

Lemma 2.5 (cf. Proposition 2.15 in [HLS09])**.** *Let $\pi\colon X \to C$ and $\pi'\colon Y \to C$ be flat projective morphisms between smooth varieties, and take a point c of C. Suppose that X_c and Y_c are the fibers of π and π' respectively over the point c,*[2] *and that we are given an integral functor $\Phi = \Phi^{\mathcal{P}}_{X \to Y}$ over C.*

Let us consider the integral functor $\Psi = \Phi^{\mathcal{P}|_{X_c \times Y_c}}_{X_c \to Y_c}$, and denote the inclusions by $k\colon X_c \hookrightarrow X$ and $k'\colon Y_c \hookrightarrow Y$. Then we have the following.

[2] Although we do not assume that X_c and Y_c are smooth, the perfectness of $\mathcal{P}|_{X_c \times Y_c}$ assures that Ψ defines a functor from $D(X_c)$ to $D(Y_c)$.

(i) Φ and Ψ satisfies $k'_* \circ \Psi \cong \Phi \circ k_*$.

(ii) Assume furthermore that Φ is an Fourier–Mukai transform. Then so is Ψ.

Proof. Assertion (i) directly follows from the projection formula and the flat base change formula (cf. [Hu06, Pages 83, 85]). For (ii), suppose that $X = Y$ and $\Phi \cong \mathrm{id}_X$. Then obviously $\Psi \cong \mathrm{id}_{X_c}$ also holds. Apply this argument to the functor $\Phi \circ \Phi^{-1}$ to get the result. ☐

2.2 The Euler form on surfaces

Let X be a smooth quasi-projective variety. For objects $\alpha, \beta \in D(X)$ with compact supports, we define the Euler form as

$$\chi(\alpha, \beta) := \sum_i (-1)^i \dim \mathrm{Hom}^i_{D(X)}(\alpha, \beta).$$

In the surface case, the Riemann-Roch theorem yields

$$\begin{aligned}\chi(\alpha, \beta) &= r(\alpha)\,\mathrm{ch}_2(\beta) - c_1(\alpha) \cdot c_1(\beta) + r(\beta)\,\mathrm{ch}_2(\alpha) \\ &\quad + \frac{1}{2}(r(\beta)c_1(\alpha) - r(\alpha)c_1(\beta)) \cdot c_1(\omega_X) + r(\alpha)r(\beta)\chi(\mathcal{O}_X).\end{aligned}$$

In particular, if $r(\alpha) = r(\beta) = 0$, we have

$$\chi(\alpha, \beta) = -c_1(\alpha) \cdot c_1(\beta). \tag{2}$$

As its application, for a (-2)-curve G on a smooth surface X, we can compute

$$\dim \mathrm{Ext}^1_X(\mathcal{O}_G(a), \mathcal{O}_G(b)) = \begin{cases} b - a - 1 & \text{if } b - a > 1 \\ 0 & \text{if } |b - a| \le 1 \\ a - b - 1 & \text{if } a - b > 1. \end{cases} \tag{3}$$

2.3 Twist functors

We introduce an important class of examples of autoequivalences.

Definition-Proposition 2.6 ([ST01]). Let X be a smooth variety, or rather a complex manifold.

(i) We say that an object $\alpha \in D(X)$ with a compact support is *spherical* if we have $\alpha \otimes \omega_X \cong \alpha$ and

$$\mathrm{Hom}^k_{D(X)}(\alpha, \alpha) \cong \begin{cases} 0 & k \neq 0, \dim X \\ \mathbb{C} & k = 0, \dim X. \end{cases}$$

(ii) Let $\alpha \in D(X)$ be a spherical object. We consider the mapping cone

$$\mathcal{C} = \mathrm{Cone}(\pi_1^*\alpha^\vee \overset{\mathbb{L}}{\otimes} \pi_2^*\alpha \to \mathcal{O}_\Delta)$$

of the natural evaluation $\pi_1^*\alpha^\vee \overset{\mathbb{L}}{\otimes} \pi_2^*\alpha \to \mathcal{O}_\Delta$, where $\Delta \subset X \times X$ is the diagonal, and π_i is the projection of $X \times X$ to the i-th factor. Then the integral functor $T_\alpha := \Phi^{\mathcal{C}}_{X\to X}$ defines an autoequivalence of the bounded derived category $D_{\mathrm{Coh}(X)}(\mathcal{O}_X - \mathrm{mod})$ of $\mathcal{O}_X$-modules with coherent cohomology on X,[3] called the *twist functor* along the spherical object α.

Remark 2.7. Let $\alpha \in D(X)$ be a spherical object. Then, by the definition, for every $\beta \in D(X)$, we have an exact triangle

$$\mathbb{R}\mathrm{Hom}_{D(X)}(\alpha, \beta) \otimes_{\mathbb{C}} \alpha \to \beta \to T_\alpha(\beta). \tag{4}$$

Suppose that $\mathrm{Supp}\,\beta \cap \mathrm{Supp}\,\alpha = \emptyset$. Then since $\mathbb{R}\mathrm{Hom}(\alpha, \beta) = 0$, we have $T_\alpha(\beta) \cong \beta$. We use this remark later. Furthermore, in the Grothendieck group $K(X)$, we have

$$[T_\alpha(\beta)] = [\beta] - \chi(\alpha, \beta)[\alpha]. \tag{5}$$

Example 2.8. (i) Let S be a smooth surface, and let G be a (-2)-curve. Then, for every integer a and any point $x \in S$, we can see

$$\chi(\mathcal{O}_G(a), \mathcal{O}_G(a)) = 2 \text{ and } \chi(\mathcal{O}_G(a), \mathcal{O}_x) = 0$$

by the equality (2). Hence, the object $\mathcal{O}_G(a) \in D(S)$ is spherical, and it follows from the equality (5) that $[T_{\mathcal{O}_G(a)}(\mathcal{O}_x)] = [\mathcal{O}_x]$.

By using (4), we can also see that

$$H^{-1}(T_{\mathcal{O}_G}(\mathcal{O}_G(2))) = \mathcal{O}_G(-1)^{\oplus 2} \text{ and } H^0(T_{\mathcal{O}_G}(\mathcal{O}_G(2))) = \mathcal{O}_G,$$

and hence $T_{\mathcal{O}_G} \notin A(S)$.

(ii) Let α be a simple coherent sheaf on an elliptic curve E, for example a line bundle or the structure sheaf $\mathcal{O}_x$ of a point $x \in E$. Then, α is spherical. Usually twist functors are not standard autoequivalences, but we can see that in this case $T_{\mathcal{O}_x} \cong \otimes\mathcal{O}_E(x) \in A(E)$ (cf. [Hu06, Example 8.10]).

(iii) Let X be a K3 surface. Then, the structure sheaf $\mathcal{O}_X$ of X is spherical. We can see that $\mathrm{Supp}\,T_{\mathcal{O}_X}(\mathcal{O}_x) = 2$ by using the triangle (4). To the contrary, we will see in Claim 3.1 that, for an elliptic surface S with non-zero Kodaira dimension, any autoequivelnce $\Phi \in \mathrm{Auteq}\,D(S)$ satisfies $\mathrm{Supp}\,\Phi(\mathcal{O}_x) \le 1$.

[3] For an algebraic variety X, or a compact complex analytic space X of dimension 2, it is known that $D_{\mathrm{Coh}(X)}(\mathcal{O}_X - \mathrm{mod})$ is equivalent to $D(X)$. See [Hu06, Corollary 3.4, Proposition 3.5] and [BV03, Corollary 5.2.2].

Theorem 2.9 (Theorem 1.3 in [IU05], Appendix A in [IUU10]). *Let X be a minimal resolution of the A_n-singularity* $\operatorname{Spec} \mathbb{C}[[x, y, z]]/(x^2 + y^2 + z^{n+1})$. *Define*

$$B := \left\langle T_{\mathcal{O}_G(a)} \mid G \text{ is a } (-2)\text{-curve} \right\rangle$$

and denote by Z the exceptional set of the resolution. Then we have

$$\operatorname{Auteq} D_Z(X) = (\langle B, \operatorname{Pic} X\rangle \rtimes \operatorname{Aut} X) \times \mathbb{Z}.$$

We will use ideas of the proof of Theorem 2.9 to prove our main result, namely Theorem 4.1.

2.4 Autoequivalences of elliptic curves

Let E be an elliptic curve. To a given $\Phi = \Phi^{\mathcal{P}} \in \operatorname{Auteq} D(E)$, we associate a group automorphism $\rho(\Phi)$ of $H^*(E, \mathbb{Z}) \cong \mathbb{Z}^4$ given by

$$\rho(\Phi^{\mathcal{P}})(-) := p_{2*}(\operatorname{ch}(\mathcal{P}) \cdot p_1^*(-))$$

(cf. [Hu06, Corollary 9.43]). This gives a group homomorphism

$$\rho\colon \operatorname{Auteq} D(E) \to \operatorname{GL}(H^*(E, \mathbb{Z})),$$

and it is known that ρ preserves the parity, i.e. it decomposes as $\rho = \eta \oplus \theta$, where $\eta(\Phi) \in \operatorname{GL}(H^{\mathrm{odd}}(E, \mathbb{Z}))$ and $\theta(\Phi) \in \operatorname{GL}(H^{\mathrm{ev}}(E, \mathbb{Z}))$.

Because $\theta(\Phi) \in \operatorname{GL}(2, \mathbb{Z})$ preserves the Euler form $\chi(\_, \_)$, we can see that $\theta(\Phi)$ actually gives an element of $\operatorname{SL}(2, \mathbb{Z})$. Take the classes $\operatorname{ch}(\mathcal{O}_E)$ and $\operatorname{ch}(\mathcal{O}_x)$ for some point x as a basis of $H^{\mathrm{ev}}(E, \mathbb{Z}) \cong \mathbb{Z}^2$. In terms of this basis, $T_{\mathcal{O}_E}$, $T_{\mathcal{O}_x} (\cong \_ \otimes \mathcal{O}_E(x))$ and $\Phi^{\mathcal{U}}$ are given by

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

respectively, where $\mathcal{U}$ is the normalized Poincare bundle on $E \times E$. Here, note that every elliptic curve is principally polarized, and hence we can identify E with $\hat{E}$. Two of these elements actually generate the group $\operatorname{SL}(2, \mathbb{Z})$, and therefore the map

$$\theta\colon \operatorname{Auteq} D(E) \to \operatorname{SL}(2, \mathbb{Z})$$

is surjective. One can compute the kernel of θ to get a short exact sequence of groups

$$1 \to \hat{E} \rtimes \operatorname{Aut} E \times \mathbb{Z}[2] \to \operatorname{Auteq} D(E) \to \operatorname{SL}(2, \mathbb{Z}) \to 1.$$

2.5 Automorphisms of elliptic surfaces

Let $\pi\colon S \to C$ and $\pi'\colon S' \to C'$ be projective elliptic surfaces, and suppose that each of S and S' has a unique elliptic fibration. Then every isomorphism $\varphi\colon S \to S'$ induces an isomorphism $C \to C'$. In the cases that we consider (namely that $S' = J_S(a,b)$; see §2.6) there is a natural identification between C and C'. Hence, the induced isomorphism is naturally regarded as an automorphism of C. We denote it by φ_C. In other words, φ_C satisfies $\pi' \circ \varphi = \varphi_C \circ \pi$. We define

$$\operatorname{Aut}_S C := \{\varphi_C \in \operatorname{Aut} C \mid \varphi \in \operatorname{Aut} S\}$$

and

$$\operatorname{Aut}_C S := \{\varphi \in \operatorname{Aut} S \mid \varphi_C = \mathrm{id}_C\}.$$

Consequently, we have a short exact sequence

$$1 \to \operatorname{Aut}_C S \to \operatorname{Aut} S \to \operatorname{Aut}_S C \to 1.$$

An elliptic surface S with $\kappa(S) \neq 0$ provides an example of a surface admitting a unique elliptic fibration: The canonical bundle formula of elliptic surfaces implies that any elliptic fibration on S is defined by the linear system $|rK_S|$ with some nonzero rational number r. Therefore, S has a unique elliptic fibration structure.

2.6 Fourier–Mukai transforms on elliptic surfaces

Bridgeland, Maciocia and Kawamata show ([BM01], [Ka02]) that for a smooth projective surface S, if S has a non-trivial Fourier–Mukai partner T, that is $|\operatorname{FM}(S)| \neq 1$, then both of S and T are abelian varieties, K3 surfaces or minimal elliptic surfaces with non-zero Kodaira dimensions.

We consider the last case in more detail. Let $\pi : S \to C$ be an elliptic surface. The results referred to in §2.6 are originally stated under the assumption that S is projective, but some of them still hold true without the projectivity of S. For our purpose, it is sometimes important to consider non-projective elliptic surfaces, hence we do not assume that S is projective unless specified otherwise.

For an object E of $D(S)$, we define the fiber degree of E as

$$d(E) = c_1(E) \cdot F,$$

where F is a general fiber of π. Let us denote by $r(E)$ the rank of E and by λ_S the highest common factor of the fiber degrees of objects of $D(S)$. Equivalently, λ_S is the smallest number d such that there is a holomorphic d-section of π. Consider integers a and b with $a > 0$ and b coprime to $a\lambda_S$. By [Br98], there exists a smooth, 2-dimensional component $J_S(a,b)$ of the moduli space of pure dimension one stable sheaves on S, the general point

of which represents a rank a, degree b stable vector bundle supported on a smooth fiber of π. There is a natural morphism $J_S(a,b) \to C$, taking a point representing a sheaf supported on the fiber $\pi^{-1}(x)$ of S to the point x. This morphism is a minimal elliptic fibration (see [Br98]). Put $J_S(b) := J_S(1,b)$. Obviously, $J_S(0) \cong J(S)$, the Jacobian surface associated to S, and $J_S(1) \cong S$. As shown in [BM01, Lemma 4.2], there is also an isomorphism

$$J_S(a,b) \cong J_S(b) \tag{6}$$

over C.

Theorem 2.10 (Theorem 5.3 in [Br98]). *Let $\pi\colon S \to C$ be an elliptic surface and take an element*

$$M = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

such that λ_S divides d and $a > 0$. Then there exists a universal sheaf $\mathcal{U}$ on $J_S(a,b) \times S$, flat over both factors, such that for any point $(x,y) \in J_S(a,b) \times S$, $\mathcal{U}|_{x\times S}$ has Chern class $(0, af, -b)$ on S and $\mathcal{U}|_{J_S(a,b)\times y}$ has Chern class $(0, af, -c)$ on $J_S(a,b)$. The resulting functor $\Phi^{\mathcal{U}}_{J_S(a,b)\to S}$ is an equivalence and satisfies

$$\begin{pmatrix} r(\Phi(E)) \\ d(\Phi(E)) \end{pmatrix} = M \begin{pmatrix} r(E) \\ d(E) \end{pmatrix} \tag{7}$$

for all objects $E \in D(J_S(a,b))$

Remark 2.11. For integers $a > 0$ and b with b coprime to $a\lambda_S$, let us consider the Fourier–Mukai transform $\Phi = \Phi^{\mathcal{U}}_{J_S(a,b)\to S}$. Take a smooth fiber F of π over a point $c \in C$, and denote by F' the smooth fiber of the morphism $J_S(a,b) \to C$ over the point c. It turns out that F' is isomorphic to $J_F(a,b)$, and hence F' is isomorphic to F by [At57, Theorem 7]. The integral transform defined by the kernel $\mathcal{U}|_{F\times F'} \in D(F \times F')$ induces an equivalence between $D(F)$ and $D(F')$ by Lemma 2.5. Fixing an isomorphism $F \cong F'$, we regard the equivalence $\Phi^{\mathcal{U}|_{F\times F'}}$ as an autoequivalence of $D(F)$.

Note that

$$M := \theta(\Phi^{\mathcal{U}|_{F\times F'}}) := \begin{pmatrix} c & a \\ d & b \end{pmatrix}$$

satisfies (7) (see §2.4 for the definition of θ). We see that λ_S divides $d = d(\Phi(\mathcal{O}_{J_S(a,b)}))$. We will use this fact in §3.4.

Theorem 2.10 implies that $J_S(b) (\cong J_S(a,b))$ is a Fourier–Mukai partner of S when $(b, \lambda_S) = 1$. Actually, the converse is also true for projective elliptic surfaces S with non-zero Kodaira dimension:

Theorem 2.12 (Proposition 4.4 in [BM01]). *Let $\pi : S \to C$ be a projective elliptic surface and S' a smooth projective variety. Assume that the Kodaira dimension $\kappa(S)$ is non-zero. Then the following are equivalent.*

(i) S' is a Fourier–Mukai partner of S.

(ii) S' is isomorphic to $J_S(b)$ for some integer b with $(b, \lambda_S) = 1$.

Remark 2.13. Theorem 2.12 tells us that the Fourier–Mukai partner S' of S has an elliptic fibration $\pi' \colon S' \to C$. Moreover, we can see $\kappa(S') = \kappa(S) \neq 0$ (cf. [BM01, Lemma 4.3]). Then, as is explained in §2.5, π' is a unique elliptic fibration structure on S'. In particular, if two elliptic surfaces are mutually Fourier–Mukai partners, the base curves of elliptic fibrations can be identified. We use this fact implicitly afterwards.

There are natural isomorphisms

$$J_S(b) \cong J_S(b + \lambda_S) \cong J_S(-b)$$

over C (see [BM01, Remark 4.5]). Therefore, we can define the subset

$$H_S := \{b \in (\mathbb{Z}/\lambda_S\mathbb{Z})^* \mid J_S(b) \cong S\}$$

of the multiplicative group $(\mathbb{Z}/\lambda_S\mathbb{Z})^*$.

Claim 2.14. *For any pair of integers b, c, with $b, c \in \mathbb{Z}/\lambda_S\mathbb{Z}$, there is an isomorphism*

$$J_{J_S(c)}(b) \cong J_S(bc).$$

Proof. Take an elliptic surface $B \to C$ with a section such that there is an isomorphism $\varphi_1 \colon J(S) \to B$ over C. Let us set $\xi := (S, \varphi_1) \in WC(B)$ (see [Ue11, §2.2] for the definition of the Weil-Chatelet group $WC(B)$). Then we see in [Ue11, §2.2] that there is an isomorphism $\varphi_c \colon J(J_S(c)) \to B$ over C such that $(J_S(c), \varphi_c)$ corresponds to the element $c\xi \in WC(B)$. By the same argument, there are isomorphisms $\varphi'_{bc} \colon J(J_{J_S(c)}(b)) \to B$ and $\varphi_{bc} \colon J(J_S(bc)) \to B$ such that both of $(J_{J_S(c)}(b), \varphi'_{bc})$ and $(J_S(bc), \varphi_{bc})$ correspond to the element $bc\xi$. This finishes the proof. $\square$

Since we have

$$J_S(1) \cong S$$

(see, e.g. [Ue11, §2.2] and [BM01, Remark 4.5]), we can see that the condition $J_S(b) \cong S$ implies that $J_S(c) \cong S$ for $c \in \mathbb{Z}$ with $bc \equiv 1 \pmod{m}$. Therefore, it turns out that H_S is a subgroup of $(\mathbb{Z}/\lambda_S\mathbb{Z})^*$. In particular, there is a natural one-to-one correspondence between the set $\mathrm{FM}(S)$ and the quotient group $(\mathbb{Z}/\lambda_S\mathbb{Z})^*/H_S$.

It is not easy to describe the group H_S concretely in general, which is equivalent to determine the set $\mathrm{FM}(S)$ (see [Ue04] and [Ue11]). However when $\lambda_S \leq 2$, $(\mathbb{Z}/\lambda_S\mathbb{Z})^*$ is trivial, and hence $\mathrm{FM}(S) = \{S\}$.

If $\lambda_S > 2$, the group H_S contains at least two elements $1, \lambda_S - 1 \in (\mathbb{Z}/\lambda_S\mathbb{Z})^*$. Hence, we have

$$|\,\mathrm{FM}(S)| \leq \varphi(\lambda_S)/2,$$

where φ is the Euler function. There are several examples in which we can compute the set $\mathrm{FM}(S)$ given in [Ue11, Example 2.6]. In the upcoming paper [Ue], the author will also give examples in which he can compute the set $\mathrm{FM}(S)$. See §7 for some more details.

Remark 2.15. Take a point $c \in C$ and an integer b with $(b, \lambda_S) = 1$. We know that there is an isomorphism $J(J_S(b)) \cong J(S)$ over the curve C (cf. [Ue11, §2.2]). Since it is known that the reduced form[4] of the fibers of S over the point c is isomorphic to the fiber of $J(S)$ over c, the same holds for the fibers of S and $J_S(b)$.

Furthermore, the multiplicities of the fibers of S and $J_S(b)$ over the same point are equal (see [Fr95, page 38] or the proof of [BM01, Lemma 4.3]). Therefore, we conclude that the fibers on S and $J_S(b)$ over the same point are isomorphic to each other.

3 Autoequivalences of elliptic surfaces with non-zero Kodaira dimension

3.1 Notation and the setting

Let

$$\pi \colon S \to C \quad \text{and} \quad \pi' \colon S' \to C$$

be projective elliptic surfaces with non-zero Kodaira dimension, and we denote the projections by

$$p \colon S \times S' \to S \quad \text{and} \quad p' \colon S \times S' \to S'.$$

Let $\Phi = \Phi^{\mathcal{P}} \colon D(S) \to D(S')$ be a Fourier-Mukai transform. The following is well-known.

Claim 3.1. *For each $y \in S$, $\operatorname{Supp} \Phi(\mathcal{O}_y)$ is contained in a single fiber of π'. Furthermore, for a point $x \in S$ with $\pi(x) \neq \pi(y)$, we have $\operatorname{Supp} \Phi(\mathcal{O}_y) \cap \operatorname{Supp} \Phi(\mathcal{O}_x) = \emptyset$.*

Proof. By the assumption on the Kodaira dimension, the cohomology class of $K_{S'}$ is a non-zero rational multiple of the cohomology class of a fiber of π'. On the other hand, since the Serre functor commutes with the equivalence Φ, there is an isomorphism

$$\Phi(\mathcal{O}_y) \cong \Phi(\mathcal{O}_y) \otimes \omega_{S'}.$$

Furthermore, since $\Phi(\mathcal{O}_y)$ is simple, we know that $\operatorname{Supp} \Phi(\mathcal{O}_y)$ is connected. These facts imply that $\operatorname{Supp} \Phi(\mathcal{O}_y)$ is contained in a single fiber of π'.

[4] If the fiber over the point c is a multiple fiber of type ${}_m\mathrm{I}_n$, then the reduced form is of type I_n.

Suppose for a contradiction that $\operatorname{Supp}\Phi(\mathcal{O}_y) \cap \operatorname{Supp}\Phi(\mathcal{O}_x) \neq \emptyset$. Take a point z in this non-empty set. Then, it follows from Lemma 2.3 that neither $\mathbb{R}\mathrm{Hom}(\Phi(\mathcal{O}_x), \mathcal{O}_z)$ nor $\mathbb{R}\mathrm{Hom}(\Phi(\mathcal{O}_y), \mathcal{O}_z)$ vanishes. Again Lemma 2.3 implies that $\operatorname{Supp}\Phi^{-1}(\mathcal{O}_z)$ contains the points x and y. This induces a contradiction, because $\operatorname{Supp}\Phi^{-1}(\mathcal{O}_z)$ is contained in a single fiber of π, and $\pi(x) \neq \pi(y)$. Therefore, we obtain $\operatorname{Supp}\Phi(\mathcal{O}_y) \cap \operatorname{Supp}\Phi(\mathcal{O}_x) = \emptyset$. □

Denote the inclusion $S' \cong y \times S' \hookrightarrow S \times S'$ by i. Notice that

$$\mathcal{P}|_{y \times S'} \cong \Phi(\mathcal{O}_y) \quad \text{and} \quad \operatorname{Supp}\mathcal{P}|_{y \times S'} = i^{-1}(\operatorname{Supp}\mathcal{P}) \tag{8}$$

(see [Hu06, Lemma 3.29]). Claim 3.1 and (8) give

$$\dim \operatorname{Supp}\mathcal{P} = 2 \text{ or } 3.$$

Claim 3.2. *Let V_1 and V_1' be non-empty open subsets of C. Let us define $U_1 := \pi^{-1}(V_1)$ and $U_1' := \pi'^{-1}(V_1')$. Suppose that there is an isomorphism $\varphi_{U_1} \colon U_1 \to U_1'$. Then there is an automorphism $\varphi_C \in \operatorname{Aut} C$ and an isomorphism $\varphi \colon S \to S'$ extending φ_{U_1} such that $\varphi_C \circ \pi = \pi' \circ \varphi$ holds.*

Proof. We may assume that $V_1' \neq C$ (otherwise the proof is done). The only proper connected subvarieties of V_1' are points. Hence $\pi'|_{U_1'} \circ \varphi_{U_1}$ maps fibers of $\pi|_{U_1}$ to points and, consequently, φ_{U_1} maps fibers to fibers. Thus there is a bijection $\varphi_{V_1} \colon V_1 \to V_1'$ such that $\varphi_{V_1} \circ \pi|_{U_1} = \pi'|_{U_1'} \circ \varphi_{U_1}$. One can deduce that φ_{V_1} is a morphism from the fact that the composition $\varphi_{V_1} \circ \pi|_{U_1}$ is a morphism.

Since C is a smooth projective curve, φ_{V_1} extends to an isomorphism φ_C. Take the elimination of indeterminancies $S \leftarrow T \to S'$ of φ_{U_1}. Then [BHPV, Proposition III.8.4] assures that the morphism $T \to C$ uniquely factors through a relatively minimal elliptic fibration. Namely, φ_{U_1} extends an isomorphism φ in the statement. □

Let us denote by Z the union of all (-2)-curves on S. Note that the set Z coincides with the union of all reducible fibers. We also denote by U the complement of Z in S, by V the image of U under π, and by F a smooth fiber of π. We define Z', U', V' and F' on S' similarly.

Suppose that there is an isomorphism $\varphi \colon S \to S'$. Then, since S and S' have a unique elliptic fibration (see §2.5), it induces an automorphism φ_C of C which satisfies $\varphi_C \circ \pi = \pi' \circ \varphi$. Moreover, an isomorphism $\varphi_U \colon U \to U'$ extends to an isomorphism $\varphi \colon S \to S'$ by Claim 3.2. In particular, we have

$$\operatorname{Aut} S \cong \operatorname{Aut} U, \quad \operatorname{Aut}_C S \cong \operatorname{Aut}_V U.$$

3.2 Autoequivalences associated with reducible fibers

In this subsection, we show that every autoequivalence of $D(S)$ induces an autoequivalence of $D_Z(S)$.

The following is crucial to show Proposition 3.4, the main result in §3.2.

Lemma 3.3. *Take a point $x \in S$.*

(i) Suppose that there is an integer i such that $H^i(\Phi(\mathcal{O}_x)) \neq 0$ and

$$c_1(H^i(\Phi(\mathcal{O}_x))) \cdot c_1(H^i(\Phi(\mathcal{O}_x))) = 0.$$

Then $\Phi(\mathcal{O}_x)$ is a shift of a sheaf.

(ii) If $x \in Z$, then $\operatorname{Supp} \Phi(\mathcal{O}_x)$ is contained in Z'.

Proof. (i) By the equation (2), we have

$$\chi(H^i(\Phi(\mathcal{O}_x)), H^i(\Phi(\mathcal{O}_x))) = -c_1(H^i(\Phi(\mathcal{O}_x))) \cdot c_1(H^i(\Phi(\mathcal{O}_x))) = 0,$$

and hence

$$\dim \operatorname{Ext}^1_{S'}(H^i(\Phi(\mathcal{O}_x)), H^i(\Phi(\mathcal{O}_x))) \geq 2.$$

But then [BM01, Lemma 2.9] implies that i is a unique integer such that $H^i(\Phi(\mathcal{O}_x)) \neq 0$. Namely $\Phi(\mathcal{O}_x)$ is a shift of sheaf.

(ii) First of all, we note that $\Phi(\mathcal{O}_y)$ is simple, because so is $\mathcal{O}_y$. First let us consider the case $\dim \operatorname{Supp} \mathcal{P} = 2$. Then there is an irreducible component W of $\operatorname{Supp} \mathcal{P}$ such that the restrictions $p|_W \colon W \to S$, $p'|_W \colon W \to S$ of projections p, p' are birational morphism (see the proof of [Ka02, Theorem 2.3]). We put

$$q := p'|_W \circ p|_W^{-1} \colon S \dashrightarrow S'.$$

But as Kawamata pointed out in [Ka02, Lemma 4.2], q is isomorphic in codimension 1, and hence in the surface case, it is an isomorphism by [Ha77, Ch.5, Lemma 5.1]. Therefore, if a (-2)-curve G contains the point x, the (-2)-curve $q(G)$ contains the point $q(x) \in \operatorname{Supp} \Phi(\mathcal{O}_x)$. Because each (-2)-curve on S' and $\operatorname{Supp} \Phi(\mathcal{O}_x)$ are always contained in a single fiber of π', $\operatorname{Supp} \Phi(\mathcal{O}_x)$ is contained in the set Z'. (Moreover we can see by [Hu06, Lemma 4.5] that $\Phi(\mathcal{O}_x) = \mathcal{O}_{q(x)}$. We use this remark below.)

Next let us consider the case $\dim \operatorname{Supp} \mathcal{P} = 3$. Suppose that $\Phi(\mathcal{O}_x)$ is a sheaf on S', after replacing Φ with $\Phi \circ [n]$ for some $n \in \mathbb{Z}$. Take a point y, which is sufficiently near the point x, but not in Z. Then $\Phi(\mathcal{O}_y)$ is also a sheaf by Lemma 2.1, and both of $\Phi(\mathcal{O}_x)$ and $\Phi(\mathcal{O}_y)$ are 1-dimensional by the assumption $\dim \operatorname{Supp} \mathcal{P} = 3$. Claim 3.1 implies that we may assume that $\Phi(\mathcal{O}_y)$ is a sheaf on a smooth elliptic curve F'. It follows from [IU05, Lemma 4.8] that $\Phi(\mathcal{O}_y)$ is a 1-dimensional simple $\mathcal{O}_{F'}$-module, and hence, it is a locally free sheaf on a smooth elliptic curve F'. Then, it is known to be stable by [Br98, Remark 3.4]. Denote the Chern class of the stable sheaf

$\Phi(\mathcal{O}_y)$ on the fiber F' by $(0, aF', -b)$ for some integer a, b. Then we know that $(a\lambda_{S'}, b) = 1$ (see the proof of [BM01, Proposition 4.4]), and hence there is the elliptic surface $J_{S'}(a,b) \to C$ together with the universal sheaf $\mathcal{U}$ on $J_{S'}(a,b) \times S'$. For the point $w \in J_{S'}(a,b)$ representing the stable sheaf $\Phi(\mathcal{O}_y)$, we have

$$\mathcal{O}_w \cong (\Phi^{\mathcal{U}}_{J_{S'}(a,b)\to S'})^{-1} \circ \Phi(\mathcal{O}_y).$$

It follows that the kernel of the Fourier–Mukai transform $(\Phi^{\mathcal{U}}_{J_{S'}(a,b)\to S'})^{-1} \circ \Phi$ has a 2-dimensional support. Now, we can apply the case $\dim \operatorname{Supp} \mathcal{P} = 2$ of the Lemma to see that there is a point z, contained in a (-2)-curve on $J_{S'}(a,b)$, such that

$$\mathcal{O}_z = (\Phi^{\mathcal{U}}_{J_{S'}(a,b)\to S'})^{-1} \circ \Phi(\mathcal{O}_x).$$

By Remark 2.15, $\operatorname{Supp} \Phi^{\mathcal{U}}_{J_{S'}(a,b)\to S'}(\mathcal{O}_z)$ is contained in the set Z'. Hence, so is $\operatorname{Supp} \Phi(\mathcal{O}_x)$.

Finally, suppose that $\dim \operatorname{Supp} \mathcal{P} = 3$ and $\Phi(\mathcal{O}_x)$ is not a shift of a sheaf. Take an integer i such that $H^i(\Phi(\mathcal{O}_x))$ is non-zero. Then the conclusion in (i) says that $c_1(H^i(\Phi(\mathcal{O}_x)))$ is not a multiple of a fiber F'. Because $H^i(\Phi(\mathcal{O}_x))$ is contained in a single fiber, this means that $\operatorname{Supp} H^i(\Phi(\mathcal{O}_x))$ is contained in a reducible fiber, and hence in Z'. $\square$

Note that, if $\operatorname{Supp} \Phi(\mathcal{O}_x)$ is contained in an irreducible fiber of π', the assumption in Lemma 3.3 (i) is satisfied by the equation (2).

Applying Lemmas 2.4 and 3.3 (ii), we obtain the following.

Proposition 3.4. *There is a natural group homomorphism*

$$\iota_Z : \operatorname{Auteq} D(S) \to \operatorname{Auteq} D_Z(S).$$

Let us define

$$\operatorname{Auteq}^{\dagger} D_Z(S) := \operatorname{Im} \iota_Z.$$

The following is used in the proof of Lemma 3.7.

Corollary 3.5. *Take a point $x \in U$. Then we have*

$$\operatorname{Supp} \Phi(\mathcal{O}_x) \subset U'.$$

Proof. We use Lemma 2.3. Take any point $y \in Z'$. Then we have

$$\mathbb{R}\mathrm{Hom}(\Phi(\mathcal{O}_x), \mathcal{O}_y) \cong \mathbb{R}\mathrm{Hom}(\mathcal{O}_x, \Phi^{-1}(\mathcal{O}_y)) = 0$$

by Lemma 3.3 (ii). This implies the assertion. $\square$

3.3 Autoequivalences associated with irreducible fibers: The classification

We begin §3.3 by classifying Fourier–Mukai transforms between elliptic surfaces without reducible fibers.

Proposition 3.6. *Let $\pi\colon S \to C$ and $\pi'\colon S' \to C$ be elliptic surfaces without reducible fibers. Here we do not assume that C is projective. Let $\Phi = \Phi^{\mathcal{P}}_{S\to S'}$ be a Fourier–Mukai transform over C such that* $\dim \operatorname{Supp} \mathcal{P} = 2$ *or* 3.

(i) The object $\mathcal{P} \in D(S \times_C S')$ is a shift of a sheaf, flat over S by the first projection.

(ii) The following are equivalent.

  (1) $\dim \operatorname{Supp} \mathcal{P} = 2$.

  (2) There are points $x \in S, y \in S'$ such that $\Phi(\mathcal{O}_x) \cong \mathcal{O}_y$.

  (3) There is a line bundle $\mathcal{L}$ on S and an isomorphism $\varphi\colon S \to S'$ over C such that $\Phi \cong \varphi_*((-) \otimes \mathcal{L})$.

(iii) Suppose that $\dim \operatorname{Supp} \mathcal{P} = 3$. *Then there are integers a, b with $(a\lambda_{S'}, b) = 1$, a universal sheaf $\mathcal{U}$ on $S' \times J_{S'}(a,b)$ and isomorphism $\phi\colon J_{S'}(a,b) \to S$ over C such that*

$$\Phi \cong \Phi^{\mathcal{U}}_{J_{S'}(a,b)\to S'} \circ \phi^*.$$

Proof. (i) Take any point $x \in S$. By the irreducibility of fibers of π' and Lemma 3.3 (i), the object $\Phi(\mathcal{O}_x)$ is a shift of a sheaf. Hence, [Br99, Lemma 4.3] implies that $\mathcal{P}$ is a shift of a sheaf, flat over S by the first projection.

(ii) As a consequence of (i), the dimension of the support of $\mathcal{P}|_{x\times S'} \cong \Phi(\mathcal{O}_x)$ does not depends on the choice of a point $x \in S$. Hence, in the situation (1) or (2), every $\Phi(\mathcal{O}_x)$ has a finite support. Then we get (3) by Lemma 2.2. Here, recall that $\mathcal{P}$ is a sheaf on $S \times_C S'$, and hence φ is defined over C. Obviously, (3) implies (1) and (2).

(iii) The proof goes parallel to that of [BM01, Proposition 4.4] (see also the proof of Lemma 3.3). Let us take a general point $x \in S$, and denote the Chern class of the stable sheaf $\Phi(\mathcal{O}_x)$ on a smooth fiber F' by $(0, aF', -b)$ for some integers a, b. Then we know that $(a\lambda_{S'}, b) = 1$, and hence we can define an elliptic surface $J_{S'}(a,b) \to C$ and a universal sheaf $\mathcal{U}$ on $S' \times J_{S'}(a,b)$. For the point $y \in J_{S'}(a,b)$ representing a stable sheaf $\Phi(\mathcal{O}_x)$, it is satisfied that

$$\Phi^{-1} \circ \Phi^{\mathcal{U}}_{J_{S'}(a,b)\to S'}(\mathcal{O}_y) \cong \mathcal{O}_x.$$

Apply Lemma 2.2, and replace a universal bundle $\mathcal{U}$ with $\mathcal{U} \otimes p^*_{J_{S'}(a,b)}\mathcal{L}$ for a line bundle $\mathcal{L}$ on $J_{S'}(a,b)$ if necessary, then we obtain the assertion. $\square$

If π has a reducible fiber, the implication from (2) to (3) in Proposition 3.6 (ii) fails because of the existence of twist functors associated to (-2)-curves.

Now we can prove the following important observation.

Lemma 3.7. *In the notation of §3.1, there is an automorphism $\delta(\Phi) \in \operatorname{Aut} C$ satisfying $\delta(\Phi)(V) = V'$ such that $\mathcal{P}|_{U \times U'}$ is a shift of a coherent sheaf on $U_{\delta(\Phi)} \times_{V'} U'$. Here $U_{\delta(\Phi)} \times_{V'} U'$ is the fiber product of U and U' over V' via the morphisms $(\delta(\Phi) \circ \pi)|_U$ and $\pi'|_{U'}$.*

Proof. Below we freely compose $\Phi = \Phi^{\mathcal{P}}$ with a shift functor if necessary. First suppose that $\dim \operatorname{Supp} \mathcal{P} = 2$. Then for every $x \in U$, Corollary 3.5 implies that $\operatorname{Supp} \Phi(\mathcal{O}_x)$ is irreducible, and then, we obtain from Lemma 3.3 (ii) and [Br99, Lemma 4.3] that $\mathcal{P}|_{U \times S'}$ is a sheaf, flat over U. In particular, the sheaf $\Phi(\mathcal{O}_x) \cong \mathcal{P}|_{x \times U'}$ has finite support. Then [Hu06, Lemma 4.5] implies that there is a point $y \in U'$ such that $\Phi(\mathcal{O}_x) \cong \mathcal{O}_y$. Then [Hu06, Corollary 6.14] implies that there is a morphism $\varphi_U \colon U \to U'$ satisfying $\Phi(\mathcal{O}_x) \cong \mathcal{O}_{\varphi_U(x)}$. Hence, we can apply Claim 3.2 to obtain the automorphism φ_C of C. Note that $\mathcal{P}|_{U \times U'}$ is the structure sheaf of the graph of the morphism φ_U, and hence a sheaf on $U_{\varphi_U} \times_{V'} U'$. This φ_C plays the role of $\delta(\Phi)$ in the assertion.

Next, we consider the case $\dim \operatorname{Supp} \mathcal{P} = 3$. By the same argument as in the proof of Proposition 3.6, there are integers a, b with $(a\lambda_{S'}, b) = 1$ such that for every point $x \in U$, there is a point $y \in J_{U'}(a, b)$ satisfying

$$\Phi^{-1} \circ \Phi^{\mathcal{U}}_{J_{S'}(a,b) \to S'}(\mathcal{O}_y) \cong \mathcal{O}_x.$$

Note that we can regard $J_{U'}(a, b)$ as the inverse image of V' by the elliptic fibration $J_{S'}(a, b) \to C$. Let us denote by $\mathcal{Q} \in D(J_{S'}(a, b) \times S)$ the kernel of $\Phi^{-1} \circ \Phi^{\mathcal{U}}_{J_{S'}(a,b) \to S'}$. Since $\dim \operatorname{Supp} \mathcal{Q} = 2$, we can apply the above argument to $\mathcal{Q}$, and then we obtain the assertion for $\mathcal{Q}|_{J_{U'}(a,b) \times U}$. Since $\mathcal{U}|_{J_{U'}(a,b) \times U'}$ is a coherent sheaf on $J_{U'}(a, b) \times_{V'} U'$, we obtain the assertion for $\mathcal{P}|_{U \times U'}$. $\square$

Remark 3.8. For a point $x \in S$, we put $c := \delta(\Phi)(\pi(x)) \in C$. Furthermore, if the point x belongs to U, we know from the definition of δ that $\operatorname{Supp} \Phi(\mathcal{O}_x)$ is contained in the fiber $\pi'^{-1}(c)$. Recall the facts that $\operatorname{Supp} \Phi(\mathcal{O}_x) = \operatorname{Supp} \mathcal{P} \cap (x \times S')$ by (8), and that $\operatorname{Supp} \Phi(\mathcal{O}_x)$ is contained in a single fiber. These facts imply that $\operatorname{Supp} \Phi(\mathcal{O}_x)$ is contained in $\pi'^{-1}(c)$ for any $x \in S$. Therefore, we conclude that $\mathcal{P}$ is an object of $D_{S_{\delta(\Phi)} \times_C S'}(S \times S')$.

On the other hand, $\mathcal{P}$ is not necessarily an object of $D(S_{\delta(\Phi)} \times_C S')$. For instance, consider a twist functor $T_{\mathcal{O}_G}$ for a (-2)-curve G on S. Because the spherical object $T_{\mathcal{O}_G}(\mathcal{O}_G(2))$ is simple, the computation in Example 2.8 (i) and [Hu06, Corollary 3.15] imply that it is not of a form $k_*\alpha$ for an object $\alpha \in D(F_c)$, a fiber F_c and the inclusion $k \colon F_c \hookrightarrow S$. Consequently, Lemma 2.5 (i) tells us that the kernel of $T_{\mathcal{O}_G}$ is not an object of $D(S \times_C S)$. Here note that $\delta(T_{\mathcal{O}_G}) = \mathrm{id}_C$.

Proposition 3.9. *In the notation of §3.1, assume furthermore that $S = S'$. Then the Fourier–Mukai autoequivalence $\Phi = \Phi^{\mathcal{P}}$ of $D(S)$ induces a Fourier–Mukai autoequivalence $\iota_U(\Phi)$ of $D(U)$ over V, by restricting the kernel $\mathcal{P}$ to $U \times U$. This ι_U defines a group homomorphism*

$$\iota_U\colon \operatorname{Auteq} D(S) \to \operatorname{Auteq} D(U)$$

satisfying that

$$\mathbb{L}i^* \circ \Phi \cong \iota_U(\Phi) \circ \mathbb{L}i^*, \tag{9}$$

where i is the inclusion $U \hookrightarrow S$.

Proof. The statement is a direct consequence of Lemma 3.7 and the isomorphisms

$$\mathrm{id}_U \cong \iota_U(\Phi \circ \Phi^{-1}) \cong \iota_U(\Phi) \circ \iota_U(\Phi^{-1}).$$

The isomorphism (9) follows from a direct computation. $\square$

Let us define

$$\operatorname{Auteq}^{\dagger} D(U) := \operatorname{Im} \iota_U.$$

Note that the elements of $\operatorname{Auteq}^{\dagger} D(U)$ are classified in Proposition 3.6.

Remark 3.10. Lemma 3.7 tells us that there is a group homomorphism

$$\delta\colon \operatorname{Auteq} D(S) \to \operatorname{Aut} C.$$

The map δ factors through the map ι_U, and hence it induces a map

$$\delta_U\colon \operatorname{Auteq}^{\dagger} D(U) \to \operatorname{Aut} V (\cong \operatorname{Aut} C).$$

Then it follows from Proposition 3.6 that

$$\begin{aligned} \operatorname{Im} \delta &= \{\varphi_C \mid \varphi \in \operatorname{Aut} S, \text{ or } \varphi\colon S \to J_S(a,b) \text{ isomorphism with } (a\lambda_S, b) = 1\} \\ &= \{\varphi_C \mid \varphi \in \operatorname{Aut} S, \text{ or } \varphi\colon S \to J_S(b) \text{ isomorphism with } (\lambda_S, b) = 1\}. \end{aligned}$$

The second equality follows from (6).

3.4 Autoequivalences associated with irreducible fibers: The structure of the group

We use the notation of §3.1 in this subsection. The aim of §3.4 is to study the structure of the group $\operatorname{Auteq}^{\dagger} D(U)$.

For $m \in \mathbb{Z}$, we define the congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ by

$$\Gamma_0(m) := \left\{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; d \in m\mathbb{Z} \right\}.$$

Let us consider the surjective map

$$\Gamma_0(\lambda_S) \to (\mathbb{Z}/\lambda_S\mathbb{Z})^*/H_S \qquad \begin{pmatrix} c & a \\ d & b \end{pmatrix} \mapsto b.$$

This is actually a group homomorphism and its kernel coincides with the group

$$\left\{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \Gamma_0(\lambda_S) \;\middle|\; J_S(b) \cong S \right\}.$$

Theorem 3.11. *There is a short exact sequence*

$$\begin{aligned} 1 \to \langle \otimes \mathcal{O}_U(D) \mid D \cdot F = 0 \rangle \rtimes \operatorname{Aut} S \times \mathbb{Z}[2] \to \operatorname{Auteq}^{\dagger} D(U) \\ \to \left\{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \Gamma_0(\lambda_S) \;\middle|\; J_S(b) \cong S \right\} \to 1. \end{aligned}$$

Proof. For $\Phi^{\mathcal{P}} \in \operatorname{Auteq}^{\dagger} D(U)$, note that $\mathcal{P}$ is a shift of a sheaf on $U_{\delta_U(\Phi)} \times_V U$ by Lemma 3.7.

Take a smooth fiber $F = F_c$ over a point $c \in V$, and denote by $F' = F_{c'}$ a smooth fiber over the point $c' = \delta(\Phi)(c)$. Let $k \colon F \hookrightarrow U$, and $k' \colon F' \hookrightarrow U$ be the natural inclusions.

It follows from Lemma 2.5 that we obtain a Fourier–Mukai transform $\Phi_{F \to F'}^{\mathcal{P}|_{F \times F'}}$ such that there is an isomorphism of functors

$$\Phi^{\mathcal{P}|_{F \times F'}} \circ \mathbb{L}k^* \cong \mathbb{L}k'^* \circ \Phi^{\mathcal{P}}.$$

By choosing a basis as in §2.4, there is an identification between $H^{\mathrm{ev}}(F, \mathbb{Z})$ and $H^{\mathrm{ev}}(F', \mathbb{Z})$. Then, we obtain a group homomorphism

$$\Theta_U \colon \operatorname{Auteq}^{\dagger} D(U) \to \mathrm{SL}(2, \mathbb{Z}).$$

Notice that this morphism does not depend on the choice of a smooth fiber F by the classification of the elements of $\operatorname{Auteq}^{\dagger} D(U)$ in Proposition 3.6.

Since F and F' are isomorphic, we fix an isomorphism. Then $\Phi_{F \to F'}^{\mathcal{P}|_{F \times F'}}$ can be regarded as an autoequivalence of $D(F)$. Then we have

$$\Theta_U(\Phi^{\mathcal{P}}) = \theta(\Phi_{F \to F'}^{\mathcal{P}|_{F \times F'}}).$$

This is important for the latter computation. (Recall the definition of θ in §2.4.)

Suppose that $\Phi = \Phi^{\mathcal{P}} \in \operatorname{Ker} \Theta_U$. Proposition 3.6 forces that $\dim \operatorname{Supp} \mathcal{P} = 2$, and hence

$$\Phi \in \langle \otimes \mathcal{O}_U(D) \mid D \cdot F = 0 \rangle \rtimes \operatorname{Aut} S \times \mathbb{Z}[2],$$

that is,

$$\operatorname{Ker} \Theta_U \cong \langle \otimes \mathcal{O}_U(D) \mid D \cdot F = 0 \rangle \rtimes \operatorname{Aut} S \times \mathbb{Z}[2].$$

Next, let us consider the image of Θ_U. Take integers a, b with $a > 0$ and $(a\lambda_S, b) = 1$. Assume that there is an isomorphism $\phi\colon J_S(a,b) \to S$. Then

$$\Phi^{\mathcal{U}}_{J_S(a,b)\to S} \circ \phi^* \tag{10}$$

gives an autoequivalence of $D(S)$. In this case, it follows from Remark 2.11 that

$$\Theta_U \circ \iota_U(\Phi^{\mathcal{U}}_{J_S(a,b)\to S} \circ \phi^*) = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

holds for some $c, d \in \mathbb{Z}$ such that λ_S divides d. We also have

$$\Theta_U \circ \iota_U(\operatorname{Pic} S) = \left\langle \begin{pmatrix} 1 & 0 \\ \lambda_S & 1 \end{pmatrix} \right\rangle$$

and

$$\Theta_U([1]) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then it follows from Proposition 3.6 that

$$\begin{aligned} &\operatorname{Im}\Theta_U \\ &= \left\langle \begin{pmatrix} 1 & 0 \\ \lambda_S & 1 \end{pmatrix}, \begin{pmatrix} c & a \\ d & b \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \;\middle|\; a \neq 0, bc - ad = 1, d \in \lambda_S\mathbb{Z}, J_S(b) \cong S \right\rangle. \end{aligned}$$

Note that

$$\begin{pmatrix} -1 & 0 \\ \lambda_S & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\lambda_S & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \operatorname{Im}\Theta_U.$$

Furthermore, we can see

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

for any $a \in \mathbb{Z}$. Therefore, we conclude that

$$\operatorname{Im}\Theta_U = \left\{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \Gamma_0(\lambda_S) \;\middle|\; J_S(b) \cong S \right\},$$

which completes the proof of Theorem 3.11. □

We define

$$\Theta\colon \operatorname{Auteq} D(S) \to \mathrm{SL}(2,\mathbb{Z})$$

to be the composition $\Theta_U \circ \iota_U$, where we regard ι_U as a surjective homomorphism from $\operatorname{Auteq} D(S)$ to $\operatorname{Auteq}^\dagger D(U)$. Thus, by definition, $\operatorname{Im}\Theta = \operatorname{Im}\Theta_U$ holds.

Remark 3.12. (i) By the remark before Theorem 3.11, we have a group isomorphism

$$\Gamma_0(\lambda_S)/\operatorname{Im}\Theta \cong (\mathbb{Z}/\lambda_S\mathbb{Z})^*/H_S.$$

As explained in §2.6, the latter group can be naturally identified with the set $\mathrm{FM}(S)$.

(ii) When $|(\mathbb{Z}/\lambda_S\mathbb{Z})^*| \leq 2$ (e.g. $\lambda_S \leq 4$), we have $(\mathbb{Z}/\lambda_S\mathbb{Z})^* = H_S$. Hence, we can see that $\operatorname{Im}\Theta = \Gamma_0(\lambda_S)$. In particular, when $\lambda_S = 1$, we see that $\operatorname{Im}\Theta = \mathrm{SL}(2,\mathbb{Z})$.

3.5 Kernels of ι_Z and ι_U

Now, let us study the kernel of the homomorphisms ι_Z given in Proposition 3.4. First of all, we may assume that $Z \neq \emptyset$ because, otherwise, we have

$$\operatorname{Ker}\iota_Z = \operatorname{Auteq} D(S) = \operatorname{Auteq}^\dagger D(U),$$

and the last group was already studied in §3.4. Take $\Phi \in \operatorname{Ker}\iota_Z$. Then for any $x \in Z$, we have $\Phi(\mathcal{O}_x) \cong \mathcal{O}_x$. Hence, by [Hu06, Corollary 6.14], there is a point $y \in U$ such that

$$\Phi(\mathcal{O}_y) \cong \mathcal{O}_w \tag{11}$$

for some $w \in U$. We can apply Proposition 3.6 (ii) for $\iota_U(\Phi)$ to obtain that, for all points $y \in U$, there is a point w satisfying (11). Therefore, Lemma 2.2 implies that $\operatorname{Ker}\iota_Z$ is contained in the group $A(S)$ of the standard autoequivalences (see §1.1), and hence

$$\operatorname{Ker}\iota_Z = \langle \otimes\mathcal{O}_S(F_c) \mid F_c \text{ a fiber of } \pi\rangle \rtimes \operatorname{Aut}_Z S, \tag{12}$$

where $\operatorname{Aut}_Z S := \{\varphi \in \operatorname{Aut} S \mid \varphi(z) = z \text{ for all } z \in Z\}$.

Let us denote

$$B := \left\langle T_{\mathcal{O}_G(a)} \mid G \text{ is a } (-2)\text{-curve } \right\rangle.$$

For the homomorphism ι_U given in Proposition 3.9, we believe that the equality

$$B = \operatorname{Ker}\iota_U \tag{13}$$

holds. Actually we have the following.

Lemma 3.13. *Suppose that equality* (13) *holds. Then Conjecture 1.1 is true.*

Proof. The result follows from the description of $\operatorname{Auteq}^\dagger D(U)$ in Theorem 3.11. □

In §4, 5 and 6, we shall check the equality (13) in the case that all reducible fibers on a given projective elliptic surface are of type I_n $(n > 1)$, and consequently show Conjecture 1.1 in this case.

4 Autoequivalences associated with singular fibers of type I_n for $n > 1$

Throughout this section, $\pi\colon S \to C$ is a projective elliptic surface whose reducible fibers are non-multiple cycles of (-2)-curves, that is, of type I_n for $n > 1$. In this case, the set Z is a disjoint union of cycles of projective lines. Below we regard Z as a closed subscheme of S equipped with the reduced induced structure.

In our setting, line bundles on (-2)-curves are spherical in the sense of Definition-Proposition 2.6. Therefore, $\operatorname{Auteq} D_Z(S)$ contains twist functors, and hence it is highly involved. The following is the main result of this article.

Theorem 4.1. *Let S be a smooth projective elliptic surface with $\kappa(S) \neq 0$. Suppose that each reducible fiber on the elliptic surface S is of type I_n for some $n > 1$. Then Conjecture 1.1 is true. Namely we have*

$$1 \to \langle B, \otimes\mathcal{O}_S(D) \mid D \cdot F = 0\rangle \rtimes \operatorname{Aut} S \times \mathbb{Z}[2] \to \operatorname{Auteq} D(S)$$
$$\overset{\Theta}{\to} \left\{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \Gamma_0(\lambda_S) \;\middle|\; J_S(a,b) \cong S \right\} \to 1.$$

Theorem 4.1 concerns the autoequivalences of the derived categories of surfaces containing $\tilde{A}_n$-configurations of (-2)-curves. Many ideas of the proof come from [IU05], where we study the autoequivalences of the derived categories of surfaces containing A_n-configurations of (-2)-curves.

4.1 Reduction of the proof of Theorem 4.1

Recall that Lemma 3.13 tells us that, if we can show the equation (13), we obtain Theorem 4.1.

Let $\{Z_j\}_{j=1}^M$ be the set of connected components of Z, that is, each Z_j is a singular fiber of type I_{n_j} for some $n_j > 1$. We define

$$B_j := \left\langle T_{\mathcal{O}_G(a)} \mid G \text{ is a } (-2)\text{-curves contained in } Z_j \right\rangle.$$

Take a connected component of Z, and denote it by Z_0. We put

$$Z_0 = C_1 \cup \cdots \cup C_n$$

such that each C_i is a (-2)-curve on S, and they satisfy

$$C_l \cdot C_m = \begin{cases} 1 & |l-m| = 1, \text{ or } |l-m| = n-1 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

in the case $n > 2$. In the case $n = 2$, C_1 and C_2 intersect each other transversely at two points.

The inclusion $B \subset \operatorname{Ker} \iota_U$ holds by Remark 2.7. Thus in order to prove (13), it is left to show $\operatorname{Ker} \iota_U \subset B$. This, in turn, can be reduced to showing the following.

Proposition 4.2 (cf. Proposition 1.7 in [IU05])**.** *Suppose that we are given an autoequivalence Φ of $D_{Z_0}(S)$ preserving the cohomology class* $\mathrm{ch}(\mathcal{O}_x) \in H^4(S, \mathbb{Q})$ *for all points $x \in Z_0$. Then, there are integers a, b $(1 \le b \le n)$ and i, and there is an autoequivalence $\Psi \in B_0$ such that*

$$\Psi \circ \Phi(\mathcal{O}_{C_1}) \cong \mathcal{O}_{C_b}(a)[i] \quad \textit{and} \quad \Psi \circ \Phi(\mathcal{O}_{C_1}(-1)) \cong \mathcal{O}_{C_b}(a-1)[i]. \tag{15}$$

Consequently, for any point $x \in C_1$, we can find a point $y \in C_b$ with $\Psi \circ \Phi(\mathcal{O}_x) \cong \mathcal{O}_y[i]$.

Suppose that we have shown Proposition 4.2. Note that $\operatorname{Ker} \iota_U \cap \operatorname{Ker} \iota_Z = \{1\}$ by (12). Thus we can consider both, $\operatorname{Ker} \iota_U$ and B, as subgroups of $\operatorname{Auteq} D_Z(S)$ and prove the inclusion $\operatorname{Ker} \iota_U \subset B$ inside this group.

Take $\Phi \in \operatorname{Ker} \iota_U$. Since $\delta(\Phi) = \mathrm{id}_C$ for δ as constructed in §3.3, Φ induces autoequivalences Φ_j of $D_{Z_j}(S)$. Since all points $x \in S$ define the same cohomology class $\mathrm{ch}(\mathcal{O}_x)$, the autoequivalence Φ_j satisfies the assumption in Proposition 4.2. We fix some j and put $n = n_j$ for simplicity. We take the irreducible decomposition of Z_j as $Z_j = C_1 \cup \cdots \cup C_n$. Now we can apply Proposition 4.2 for Φ_j to find $\Psi_j \in B_j$ satisfing (15). Since $\Psi_j \circ \Phi$ also belongs to $\operatorname{Ker} \iota_U$, we have $b = 1$ and $i = 0$, and $x = y$ in Proposition 4.2. Hence, $\Psi_j \circ \Phi_j$ gives an autoequivalence of $D_{Z'_j}(S)$, where $Z'_j = C_2 \cup \cdots \cup C_n$. Since Z'_j is an A_{n-1}-configuration of (-2)-curves, [IU05, Theorem 1.3] implies that

$$\begin{aligned} \Psi_j \circ \Phi_j \in & ((\left\langle B'_j, \operatorname{Pic} S \right\rangle \rtimes \operatorname{Aut} S) \times \mathbb{Z}) \cap \operatorname{Ker} \iota_U \\ \cong & B'_j \rtimes \langle \otimes \mathcal{O}_S(C_i) \mid i = 1, \ldots, n \rangle \\ \subset & B_j, \end{aligned}$$

where we put $B'_j := \left\langle T_{\mathcal{O}_{C_i}(a)} \mid i = 2, \ldots, n \right\rangle$ and the last inclusion is a consequence of [IU05, Proposition 4.18 (i)]. Hence, we know that $\Phi_j \in B_j$.

We apply this argument for each Z_j $(1 \le j \le M)$, and then we can see that

$$\Psi_1 \circ \cdots \circ \Psi_M \circ \Phi \in B$$

and hence $\Phi \in B$. Therefore, we have $\operatorname{Ker} \iota_U \subset B$ as desired.

Remark 4.3. In the A_n-case of Proposition 4.2, that is [IU05, Proposition 1.7], we do not need the assumption that Φ preserves the cohomology class $\mathrm{ch}(\mathcal{O}_x)$, since it is always true. To the contrary, in the $\tilde{A}_n$-case, the image of $\Phi^{\mathcal{U}}_{J_S(a,b) \to S} \circ \phi^*$ in (10) under ι_Z does not preserve the cohomology class $\mathrm{ch}(\mathcal{O}_x)$. The existence of such elements forces us to put this assumption in Proposition 4.2.

The subgroup

$$\langle B, (\operatorname{Pic} S/\langle \otimes \mathcal{O}_S(F_c) \mid c \in V\rangle) \rtimes (\operatorname{Aut} S/\operatorname{Aut}_Z S) \times \mathbb{Z}[1]\rangle$$

of $\operatorname{Auteq}^\dagger D_Z(S)$ preserves $H^4(S, \mathbb{Q})$, and therefore it is strictly smaller than $\operatorname{Auteq}^\dagger D_Z(S)$. This is in contrast to the A_n-case of Theorem 2.9.

We shall prove Proposition 4.2 in §6. As an intermediate step, we first show the following.

Proposition 4.4 (cf. Proposition 1.6 in [IU05]). *Let α be a spherical object in $D_{Z_0}(S)$. Then there are integers a, b $(1 \le b \le n)$ and i, and there is an autoequivalence $\Psi \in B_0$ such that*

$$\Psi(\alpha) \cong \mathcal{O}_{C_b}(a)[i].$$

Proposition 4.4 is proved in §5.

4.2 The cohomology sheaves of spherical objects

Torsion free sheaves on a chain of projective lines We consider the cycle of lines $Z_0 = C_1 \cup \cdots \cup C_n$ as an abstract variety and denote it by $\tilde{\mathbb{I}}_n$. The curves C_i's are labelled as in (14) when $n > 2$.

We denote by $\mathbb{I}_n$ a chain of n projective lines. We put $\mathbb{I}_n = C'_1 \cup \cdots \cup C'_n$ such that each C'_i is a projective line, and they satisfy

$$C'_l \cdot C'_m = \begin{cases} 1 & |l - m| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For a coherent sheaf $\mathcal{R}$ on $\mathbb{I}_n$ or $\tilde{\mathbb{I}}_n$, we denote by $\deg_C \mathcal{R}$ the degree of the restriction $\mathcal{R}|_C$ to the component C of $\mathbb{I}_n$ or $\tilde{\mathbb{I}}_n$. It is known that a line bundle $\mathcal{L}$ on $\mathbb{I}_n$ is determined by the degree $\mathcal{L}|_C$ on all the components C, that is,

$$\operatorname{Pic} \mathbb{I}_n \cong \mathbb{Z}^n.$$

The line bundle corresponding to the vector $(a_1, \dots, a_n) \in \mathbb{Z}^n$ is denoted by

$$\mathcal{O}_{\mathbb{I}_n}(a_1, \dots, a_n).$$

When we write $*$ instead of a_l, we do not specify the degree at C'_l. For instance, when we write

$$\mathcal{R}_1 = \mathcal{O}_{\mathbb{I}_3}(a, b, *),$$

this means that $\mathcal{R}_1$ is a line bundle on $\mathbb{I}_3$ such that $\deg_{C'_1} \mathcal{R}_1 = a$, $\deg_{C'_2} \mathcal{R}_1 = b$ and $\deg_{C'_3} \mathcal{R}_1$ is arbitrary. The expression

$$\mathcal{R}_2 = \mathcal{O}_{C'_1 \cup \cdots}(a, *)$$

means that $\mathcal{R}_2 = \mathcal{O}_{\mathbb{I}_k}(a, *, \dots, *)$ for some (not further specified) $k \geq 2$. Note that the support of $\mathcal{R}_2$ is strictly larger than C_1'. We often use figures

$$\begin{array}{rl} & \begin{array}{ccccc} C_1' & & C_2' & & C_3' \end{array} \\ \mathcal{R}_1 : & \textcircled{\scriptsize a}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\textcircled{\scriptsize b}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bigcirc \\ \mathcal{R}_2 : & \textcircled{\scriptsize a}\!\!-\!\!\!-\!\!\!-\!\!\!- \end{array} \tag{16}$$

to define $\mathcal{R}_1, \mathcal{R}_2$ above. We use a dotted line

$$\begin{array}{rl} & C_1' \\ \mathcal{R}_3 : & \textcircled{\scriptsize a}\!- - - \end{array} \tag{17}$$

to indicate that $\mathcal{R}_3$ is either $\mathcal{O}_{C_1'}(a)$ or $\mathcal{O}_{C_1' \cup \cdots}(a, *)$.

Torsion free, but not locally free sheaves on $\tilde{\mathbb{I}}_n$ For any $m \in \mathbb{Z}$ satisfying $m - i \in n\mathbb{Z}$ with some $1 \leq i \leq n$, we define

$$C_m := C_i.$$

Proposition 4.5 (Theorem 19 in [BBDG])**.** *If an indecomposable torsion free $\mathcal{O}_{\tilde{\mathbb{I}}_n}$-module $\mathcal{S}$ is not locally free, then there is a finite surjective morphism*

$$p_k \colon \mathbb{I}_k \to \tilde{\mathbb{I}}_n,$$

some integer s, and a line bundle $\mathcal{L}$ on $\mathbb{I}_k$ such that $p_k(C_l') = C_{l+s-1}$ with $l = 1, \dots, k$, and

$$\mathcal{S} \cong p_{k*}\mathcal{L}.$$

In the situation of Proposition 4.5, assume that

$$\mathcal{L} \cong \mathcal{O}_{\mathbb{I}_k}(a_s, \dots, a_{s+k-1}).$$

In this case, we set

$$\mathcal{S}_s(a_s, \dots, a_{s+k-1}) := \mathcal{S}, \quad \text{or} \quad \mathcal{S}_{C_s \cup \cdots \cup C_{s+k-1}}(a_s, \dots, a_{s+k-1}) := \mathcal{S}.$$

We can see that

$$\mathcal{S}_s(a_s, \dots, a_{s+k-1})|_{C_m} \cong \bigoplus_{l \in m + n\mathbb{Z}, s \leq l \leq s+k-1} \mathcal{O}_{C_m}(a_l). \tag{18}$$

Notice that, for $k < n$, we have

$$\mathcal{S}_{C_1 \cup \cdots \cup C_k}(0, \dots, 0) = \mathcal{S}_1(\overbrace{0, \dots, 0}^{k}) \cong \mathcal{O}_{C_1 \cup \cdots \cup C_k},$$

but in contrast

$$\mathcal{S}_{C_1 \cup \cdots \cup C_n}(0, \dots, 0) = \mathcal{S}_1(\overbrace{0, \dots, 0}^{n}) \not\cong \mathcal{O}_{\tilde{\mathbb{I}}_n}.$$

The cohomology sheaves of spherical objects Henceforth, we freely use the notations and results on $\widetilde{\mathbb{I}}_n$ mentioned above.

For a complex analytic open subset U of S and a spherical object $\alpha \in D(S)$, let

$$\Sigma(\alpha)_U$$

be the set of all indecomposable summands of the coherent $\mathcal{O}_U$-module $\bigoplus_p H^p(\alpha)|_U$. If $U = S$, we just denote it by

$$\Sigma(\alpha).$$

We frequently use the following.

Lemma 4.6. *(i) For a spherical object $\alpha \in D_{Z_0}(S)$, the direct sum $\bigoplus_p H^p(\alpha)$ of its cohomology sheaves is rigid as an $\mathcal{O}_S$-module, and a torsion free $\mathcal{O}_{Z_0}$-module.*

(ii) Any $\mathcal{R} \in \Sigma(\alpha)$ cannot be a locally free $\mathcal{O}_{Z_0}$-module, and it is of the form $\mathcal{S}_s(a_s, a_{s+1}, \dots, a_t)$ for some integers i, j with $C_{s-1} \neq C_t$.

Proof. (i) This is a direct consequence of [IU05, Proposition 4.5, Lemmas 4.8, 4.9].

(ii) For a torsion free sheaf $\mathcal{E}$ on Z_0 such that $c_1(\mathcal{E})$ is some multiple of $[Z_0]$, we can see that $\chi_S(\mathcal{E}, \mathcal{E}) = -c_1(\mathcal{E})^2 = 0$, which implies that $\mathcal{E}$ is not rigid. Hence, the torsion freeness and the rigidity of $\mathcal{R}$ by (i) imply the result. □

Remark 4.7. Suppose that S has a multiple fiber mZ_0, i.e. of type ${}_m\mathrm{I}_n$ for some $m > 1, n > 0$. Then [IU05, Lemma 4.8] implies that $\bigoplus_p H^p(\alpha)$ is an $\mathcal{O}_{mZ_0}$-module. In particular, we cannot conclude Lemma 4.6 (i). This is the reason why we assume that each reducible fiber is non-multiple in Theorem 1.3.

For a connected union of (-2)-curves

$$Z' := C_s \cup C_{s+1} \cup \dots \cup C_t$$

contained in Z_0, take a sufficiently small complex analytic neighbourhood of Z', and we denote it by

$$U_{s,\dots,t}.$$

Here, sufficiently small means that

$$Z_0 \cap U_{s,\dots,t} \cong D_{s-1} \cup C_s \cup \dots \cup C_t \cup D_{t+1}$$

where D_i is a one-dimensional complex disk.

For a given $\beta \in D_{Z_0}(S)$, define

$$l_i(\beta) := \sum_p \mathrm{length}_{\mathcal{O}_{S,\eta_i}} H^p(\beta)_{\eta_i}$$

for each curve C_i, where $\mathcal{O}_{S,\eta_i}$ is the local ring of S at the generic point η_i of C_i, $H^p(\beta)_{\eta_i}$ is the stalk over η_i and $\mathrm{length}_{\mathcal{O}_{S,\eta_i}}$ measures the length over $\mathcal{O}_{S,\eta_i}$. Furthermore we define

$$l(\beta) := \sum_{i=1}^{n} l_i(\beta),$$

and also define

$$l(\beta|_U) := \sum_{i\colon U\cap C_i \neq \emptyset} l_i(\beta)$$

for a complex analytic open subset U of S. These invariants play important roles in the induction step in the proofs of Propositions 4.2 and 4.4.

Using Lemma 4.6, we can deduce strong conditions on the elements of $\Sigma(\alpha)$ as Lemma 4.9 below. Before giving a general statement in Lemma 4.9, we first give a special one for the case $n = 2$, since the proof is slightly different from the one in the case $n > 2$.

Lemma 4.8. *Let us consider the case $n = 2$, i.e. $Z_0 \cong \tilde{\mathbb{I}}_2$. and let α be a spherical object in $D_{Z_0}(S)$. Assume that there is an element*

$$\mathcal{S} := \mathcal{S}_s(a_s, a_{s+1}, \ldots, a_t) \in \Sigma(\alpha)$$

with $s \neq t$, which means that $l(\mathcal{S}) > 1$. Then there are integers a and $i = 1$ or 2 satisfying the following conditions:

(i) We have $C_i = C_s = C_t$ and $a = a_s = a_t$. The integer i (resp. The integer a) does not depend on the choice of $\mathcal{S} \in \Sigma(\alpha)$ (resp. $\mathcal{S} \in \Sigma(\alpha)$ with $l(\mathcal{S}) > 1$).

(ii) Assume that there is an element $\mathcal{S}' \in \Sigma(\alpha)$ with $l(\mathcal{S}') = 1$. Then $\mathcal{S}' = \mathcal{O}_{C_i}(b)$ with $b = a$ or $a - 1$.

(iii) In the situation of (ii), assume furthermore that the above $\mathcal{S}$ satisfies $l(\mathcal{S}) > 3$. Then we have

$$a_{s+2} = a_{s+4} = \cdots = a_{t-2} = b + 1.$$

Proof. Take elements

$$\mathcal{S}_1 := \mathcal{S}_{s_1}(a_{s_1}, \ldots, a_{t_1}), \quad \mathcal{S}_2 := \mathcal{S}_{s_2}(b_{s_2}, \ldots, b_{t_2}) \in \Sigma(\alpha).$$

First, we show $C_{s_1} = C_{t_2}$. To the contrary, suppose that $C_{s_1} \neq C_{t_2}$, namely $C_{s_1} = C_{t_2+1}$ holds. Then there is a non-split exact sequence

$$0 \to \mathcal{S}_2 \to \mathcal{S}_{s_2}(b_{s_2}, \ldots, b_{t_2-1}, b_{t_2} + 1, a_{s_1}, \ldots, a_{t_1}) \to \mathcal{S}_1 \to 0,$$

which gives a contradiction with the rigidity of $\mathcal{S}_1 \oplus \mathcal{S}_2$ in Lemma 4.6 (i). Thus, we obtain $C_{s_1} = C_{t_2}$.

If we replace the role of $\mathcal{S}_1$ and $\mathcal{S}_2$ in the above argument, then we obtain $C_{s_2} = C_{t_1}$. Furthermore, consider the special case $\mathcal{S}_1 = \mathcal{S}_2$. Then this particularly implies the equalities

$$C_{s_1} = C_{t_1} = C_{s_2} = C_{t_2}. \tag{19}$$

Hence, we obtain the assertion $C_i = C_s = C_t$ in (i), and the assertion $\operatorname{Supp}\mathcal{S}' = C_i$ in (iii). The equalities (19) and the rigidity of $\mathcal{S}_1 \oplus \mathcal{S}_2$ also imply that

$$2 = -c_1(\mathcal{S}_1)c_1(\mathcal{S}_2) = \chi_S(\mathcal{S}_1, \mathcal{S}_2) = \dim \operatorname{Hom}_S(\mathcal{S}_1, \mathcal{S}_2) + \dim \operatorname{Ext}^2_S(\mathcal{S}_1, \mathcal{S}_2). \tag{20}$$

Note that $\operatorname{Ext}^2_S(\mathcal{S}_1, \mathcal{S}_2) \cong \operatorname{Hom}_S(\mathcal{S}_2, \mathcal{S}_1)^\vee$ by the Grothendieck–Serre duality.

(i) Now suppose that $l(\mathcal{S}_1) > 1$ and $l(\mathcal{S}_2) > 1$. If $a_{s_1} > a_{t_1}$, there is a morphism

$$\mathcal{S}_1 \twoheadrightarrow \mathcal{O}_{C_{t_1}}(a_{t_1}) = \mathcal{O}_{C_{s_1}}(a_{t_1}) \hookrightarrow \mathcal{S}_1.$$

And hence, we have $\dim \operatorname{Hom}_S(\mathcal{S}_1, \mathcal{S}_1) \geq 2$, which contradicts (20) in the case $\mathcal{S}_1 = \mathcal{S}_2$. Then, we conclude that $a_{s_1} = a_{t_1}$ and $b_{s_2} = b_{t_2}$.

Furthermore, if $a_{s_1} \neq b_{s_2}$, we see that

$$\dim \operatorname{Hom}_S(\mathcal{S}_1, \mathcal{S}_2) \geq 4 \text{ or } \dim \operatorname{Ext}^2_S(\mathcal{S}_1, \mathcal{S}_2) \geq 4,$$

which gives a contradiction with (20).

(ii) Assume that $b > a$. Then $\dim \operatorname{Hom}_S(\mathcal{S}, \mathcal{S}') \geq 4$. Similarly, if $b < a - 1$, we have $\dim \operatorname{Hom}_S(\mathcal{S}', \mathcal{S}) \geq 4$. Hence, both possibilities contradict (20), and we conclude that $b = a$ or $a - 1$ as asserted.

(iii) Suppose that $b = a$ in (ii). Then by (20) we see

$$\dim \operatorname{Hom}_S(\mathcal{S}, \mathcal{S}') = 2 \text{ and } \dim \operatorname{Hom}_S(\mathcal{S}', \mathcal{S}) = 0.$$

This impies the conclusion. Next, suppose that $b = a - 1$. Then, (20) implies

$$\dim \operatorname{Hom}_S(\mathcal{S}, \mathcal{S}') = 0 \text{ and } \dim \operatorname{Hom}_S(\mathcal{S}', \mathcal{S}) = 2.$$

This implies the conclusion. □

Let us proceed a general statement for any n.

Lemma 4.9. *Let α be a spherical object in $D_{Z_0}(S)$. Take elements*

$$\mathcal{S}_1 := \mathcal{S}_{s_1}(a_{s_1}, \dots, a_{t_1}), \quad \mathcal{S}_2 := \mathcal{S}_{s_2}(b_{s_2}, \dots, b_{t_2}) \in \Sigma(\alpha).$$

(i) Let us take a reduced closed subscheme $Z' = C_i \cup C_{i+1} \cup \cdots \cup C_j$ of Z_0 for some $i, j \in \mathbb{Z}$ with $0 \leq j - i \leq n - 1$. Then $(\mathcal{S}_1 \oplus \mathcal{S}_2)|_{Z'}$ is a rigid $\mathcal{O}_S$-module.

(ii) *We have* $C_{t_1} \neq C_{s_2-1}$.

(iii) *For integers* l, m *satisfying* $s_1 \leq l \leq t_1$ *and* $s_2 \leq m \leq t_2$ *such that* $C_l = C_m$, *we have the following.*

(1) $|a_l - b_m| \leq 1$.

(2) *Suppose that* $s_1 < l \leq t_1$ *and* $s_2 = m \leq t_2$*:*

$$
\begin{array}{rccc}
 & C_{l-1} & C_l & C_{l+1} \\
\textit{A part of } \mathcal{S}_1 : & \text{———} & a_l & \text{- - - -} \\
\textit{The beginning of } \mathcal{S}_2 : & & b_m & \text{- - - -}
\end{array}
$$

Then we have $a_l \geq b_m$.

(3) *Suppose that* $s_1 < l < t_1$ *and* $s_2 = m = t_2$*:*

$$
\begin{array}{rccc}
 & C_{l-1} & C_l & C_{l+1} \\
\textit{A part of } \mathcal{S}_1 : & \text{———} & a_l & \text{———} \\
\mathcal{S}_2 : & & b_m &
\end{array}
$$

Then we have $a_l = b_m + 1$.

(4) *Suppose that* $s_1 < l = t_1$ *and* $s_2 = m < t_2$.

$$
\begin{array}{rccc}
 & C_{l-1} & C_l & C_{l+1} \\
\textit{The end of } \mathcal{S}_1 : & \text{———} & a_l & \\
\textit{The beginning of } \mathcal{S}_2 : & & b_m & \text{———}
\end{array}
$$

Then we have $a_l = b_m$.

Proof. (i) We may assume that $Z' \neq Z$. Let us consider the restriction map

$$S_1 \oplus \mathcal{S}_2 \to (\mathcal{S}_1 \oplus \mathcal{S}_2)|_{Z'}.$$

Note that there are no homomorphism from its kernel to $(\mathcal{S}_1 \oplus \mathcal{S}_2)|_{Z'}$, since their supports intersect at finitely many points. Then we can apply Mukai's lemma (see [KO95, Lemma 2.2 (2)]), and hence the rigidity of $\mathcal{S}_1 \oplus \mathcal{S}_2$ implies the rigidity of $(\mathcal{S}_1 \oplus \mathcal{S}_2)|_{Z'}$ as an $\mathcal{O}_S$-module.

(ii) To the contrary, assume that $C_{t_1} = C_{s_2-1}$. Then we can actually deduce a contradiction in a completely similar way to the one in the proof of $C_s = C_t$ in Lemma 4.8 (i).

(iii) Take $Z' = C_l$ in (i) for (1), and $Z' = C_{l-1} \cup C_l$ in (i) for (2) in the case $n > 2$. Then (1) follows immediately from the equation (3) in §2.2 and the rigidity of

$$\mathcal{O}_{C_l}(a_l) \oplus \mathcal{O}_{C_m}(b_m) = \mathcal{O}_{C_l}(a_l) \oplus \mathcal{O}_{C_l}(b_m),$$

which is a direct summand of the rigid sheaf $(\mathcal{S}_1 \oplus \mathcal{S}_2)|_{C_l}$. In the situation of (2) in the case $n > 2$, we know the rigidity of

$$\mathcal{O}_{C_{l-1}\cup C_l}(a_{l-1}, a_l) \oplus \mathcal{O}_{C_m}(b_m) = \mathcal{O}_{C_{l-1}\cup C_l}(a_{l-1}, a_l) \oplus \mathcal{O}_{C_l}(b_m).$$

Then, there is a surjection

$$\mathrm{Hom}_S(\mathcal{O}_{C_l}(b_m), \mathcal{O}_{C_l}(a_l)) \to \mathrm{Ext}^1_S(\mathcal{O}_{C_l}(b_m), \mathcal{O}_{C_{l-1}}(a_{l-1} - 1)).$$

The non-vanishing of the latter space forces the result. The statement of (2) in the case $n = 2$ is proved in Lemma 4.8.

We leave it to the reader to show (3) since all the ideas have already appeared. The statement (4) is a direct consequence of (2). $\square$

5 The proof of Proposition 4.4

Suppose that we are given a spherical object $\alpha \in D_{Z_0}(S)$ with $l(\alpha) = 1$. Then we get $\alpha \cong \mathcal{O}_{C_b}(a)[i]$ for some $a, b, i \in \mathbb{Z}$. (Or, assume that $l(\alpha) < n$. Then $\mathrm{Supp}\,\alpha \neq Z_0$, and then [IU05, Proposition 1.6] implies Proposition 4.4.) If we prove that for a spherical α with $l(\alpha) > 1$, there is an autoequivalence $\Psi \in B_0$ such that

$$l(\Psi(\alpha)) < l(\alpha), \tag{21}$$

then, since $\Psi(\alpha)$ is again spherical, the induction on $l(\alpha)$ yields Proposition 4.4. On the other hand, we can show the inequality

$$l(\Psi(\alpha)) \leq \sum_q l(\Psi(H^q(\alpha)))$$

by a similar way to [IU05, Lemma 4.11] for any $\Psi \in \mathrm{Auteq}\, D_{Z_0}(S)$. Thus to get (21), it is enough to show that

$$\sum_q l(\Psi(H^q(\alpha))) < \sum_q l(H^q(\alpha))(= l(\alpha)). \tag{22}$$

5.1 Auxiliary results

Let us begin the following lemma.

Lemma 5.1. *Take integers s, t with $s \leq t$, a complex analytic open subset $U = U_{s,\dots,t}$ and $\Psi \in \left\langle T_{\mathcal{O}_{C_i}(a)} \mid s \leq i \leq t, a \in \mathbb{Z} \right\rangle$. Let β be an object of $D_{Z_0}(S)$. Then we have the following:*

(i) $\Psi(\beta)|_U \cong \Psi(\beta|_U)$, *and*

(ii) $l(\Psi(\beta)) - l(\beta) = l(\Psi(\beta)|_U) - l(\beta|_U)$.

Proof. It is enough to consider the case $\Psi = T_{\mathcal{O}_C(a)}$ for some a and some $C = C_i$ with $s \leq i \leq t$.

(i) Since $\operatorname{Supp} \mathcal{O}_C(a) = C \subset U$, we have

$$\mathbb{R}\mathrm{Hom}_{D(S)}(\mathcal{O}_C(a), \beta) \cong \mathbb{R}\mathrm{Hom}_{D(U)}(\mathcal{O}_C(a), \beta|_U).$$

Hence, there is the isomorphism of exact triangles in D(U):

$$\begin{tikzcd}
\mathbb{R}\mathrm{Hom}_{D(S)}(\mathcal{O}_C(a), \beta) \otimes_{\mathbb{C}} \mathcal{O}_C(a) \arrow[r] \arrow[d, "\cong"] & \beta|_U \arrow[r] \arrow[d, equal] & T_{\mathcal{O}_C(a)}(\beta)|_U \arrow[d, "\cong"] \\
\mathbb{R}\mathrm{Hom}_{D(U)}(\mathcal{O}_C(a), \beta|_U) \otimes_{\mathbb{C}} \mathcal{O}_C(a) \arrow[r] & \beta|_U \arrow[r] & T_{\mathcal{O}_C(a)}(\beta|_U)
\end{tikzcd}$$

(ii) From the exact triangle (4), it is easy to see the equality

$$(T_{\mathcal{O}_C(a)}(\beta))|_{S \setminus C} = \beta|_{S \setminus C}.$$

Hence we have

$$\sum_{i\colon U \cap C_i = \emptyset} l_i(T_{\mathcal{O}_C(a)}(\beta)) = \sum_{i\colon U \cap C_i = \emptyset} l_i(\beta).$$

Since by definition of $l(\beta|_U)$ we have

$$l(\beta) = l(\beta|_U) + \sum_{i\colon U \cap C_i = \emptyset} l_i(\beta),$$

we obtain

$$l(T_{\mathcal{O}_C(a)}(\beta)) - l(\beta) = l((T_{\mathcal{O}_C(a)}(\beta))|_U) - l(\beta|_U).$$

as desired $\square$

By Lemma 5.1 (i), we can use the computation in [IU05, Lemma 4.15] in our setting. For example, when $n = 3$, taking $U = U_{0,1}$ and $U_{1,2}$ in Lemma 5.1 (i), we have $T_{\mathcal{O}_{C_1}(-1)}(\mathcal{S}_{C_1 \cup C_2 \cup C_3 \cup C_1}(0,0,0,0)) = \mathcal{O}_{C_2 \cup C_3}$.

Lemma 5.1 (ii) is useful to prove (22) as it allows to apply many results from the A_n-case of [IU05] to our $\tilde{A}_n$-case. Namely, let $\mathcal{S} \in \Sigma(\alpha)$ and $U = U_{s,\dots,t}$ as above. Then there is a smooth surfaces $\hat{S}$ containing an A_{t-s+3}-configuration $\hat{Z}_0$ and an open subset $\hat{U} \cong U$ such that $\hat{U} \cap \hat{Z}_0 \cong U \cap Z$ together with a sheaf $\hat{\mathcal{S}}$ on $\hat{S}$ such that $\hat{\mathcal{S}}|_{\hat{U}} \cong \mathcal{S}|_U$. Now applying Lemma 5.1 (ii) twice, we get the equality

$$l(\Psi(\mathcal{S})) - l(\mathcal{S}) = l(\Psi(\hat{\mathcal{S}})) - l(\hat{\mathcal{S}}).$$

The right hand side of this equation is computed in many cases in [IU05], see in particular [IU05, Lemma 4.15]. In the following, we will often refer

the reader to statements in [IU05] and claim that the proof in our case is analogous. Note that in many steps of the proof in [IU05], we referred to [IU05, (6.2)] and left the details to the reader. Many of the details (or rather their analogues in the $\tilde{A}_n$-case) are stated explicitly in Lemmas 4.8 and 4.9 of the present paper.

To prove Lemma A below, we need local versions of [IU05, Lemmas 6.3, 6.6];

Lemma 5.2 (cf. Lemma 6.3 in [IU05]). *Let $\alpha \in D_{Z_0}(S)$ be a spherical object and $C = C_s \subset Z_0$ a (-2)-curve. Take a sufficiently small complex analytic neighbourhood $U = U_s$ of C. Assume that for every p we have a decomposition*

$$H^p(\alpha)|_U = \bigoplus_j^{r_1^p} \mathcal{R}^p_{1,j} \oplus \bigoplus_j^{r_2^p} \mathcal{R}^p_{2,j} \oplus \bigoplus_j^{r_3^p} \mathcal{R}^p_{3,j} \oplus \bigoplus_j^{r_4^p} \mathcal{R}^p_{4,j},$$

where $\mathcal{R}^p_{k,j}$'s are sheaves of the forms:

$$\begin{array}{lc} & C \\ \mathcal{R}^p_{1,j}: & \text{——}\textcircled{0}\text{——} \\ \mathcal{R}^p_{2,j}: & \phantom{\text{——}}\textcircled{0}\text{——} \\ \mathcal{R}^p_{3,j}: & \phantom{\text{——}}\textcircled{-1}\text{——} \\ \mathcal{R}^p_{4,j}: & \phantom{\text{——}}\textcircled{-1}\phantom{\text{——}} \end{array}$$

In this situation, we have the following:

(i) If $\sum_p r_2^p > \sum_p r_3^p$, then $l(T_{\mathcal{O}_C(-1)}(\alpha)) < l(\alpha)$.

(ii) If $\sum_p r_2^p < \sum_p r_3^p$, then $l(T_{\mathcal{O}_C(-2)}(\alpha)) < l(\alpha)$.

Proof. The assumption in (i) and [IU05, Lemma 4.15] imply the inequality

$$\sum_p l(T_{\mathcal{O}_C(-1)}(H^p(\alpha)|_U)) < \sum_p l(H^p(\alpha)|_U)(= l(\alpha|_U)).$$

Combining this with Lemma 5.1, we obtain the inequality (22) for $\Psi = T_{\mathcal{O}_C(-1)}$. Then the desired result follows as explained above.

The proof of (ii) is similar. $\square$

Lemma 5.3 (cf. Lemma 6.6 in [IU05]). *Under the assumptions of Lemma 5.2, assume that $\sum_p r_2^p = \sum_p r_3^p \neq 0$ holds. Then $r_4^p = 0$ for all p.*

Proof. According to the decomposition of $H^p(\alpha)|_U$ in Lemma 5.2, we can decompose $H^p(\alpha)$ as

$$H^p(\alpha) = \bigoplus_{j'} \widetilde{\mathcal{R}^p_{1,j'}} \oplus \bigoplus_{j}^{r_2^p} \widetilde{\mathcal{R}^p_{2,j}} \oplus \bigoplus_{j}^{r_3^p} \widetilde{\mathcal{R}^p_{3,j}} \oplus \bigoplus_{j}^{r_4^p} \mathcal{R}^p_{4,j},$$

where $\widetilde{\mathcal{R}^p_{1,j'}}$ satisfies $s(\widetilde{\mathcal{R}^p_{1,j'}}) \neq s$, and $\widetilde{\mathcal{R}^p_{k,j}}$ $(k = 2, 3)$ is a sheaf in $\Sigma(\alpha)$ such that $\mathcal{R}^p_{k,j}$ is a direct summand of $\widetilde{\mathcal{R}^p_{k,j}}|_U$. Here note that $\widetilde{\mathcal{R}^p_{2,j}}|_U$ and $\widetilde{\mathcal{R}^p_{3,j}}|_U$ may possibly contain several direct summands of the form $\mathcal{R}^p_{1,j}$.

Apply the proof of [IU05, Lemma 6.6] for this decomposition, then we obtain the assertion. Note that in the proof we use [IU05, Lemmas 6.4, 6.5]. However, our setting requires some slight changes. For example, we should replace the notations X, Z with S, Z_0 respectively, and the vertical arrows of the diagram [IU05, (6.3)] are not isomorphism any more, but injective. Hence, we define $\bar{\eta}^p$ to be the following composite:

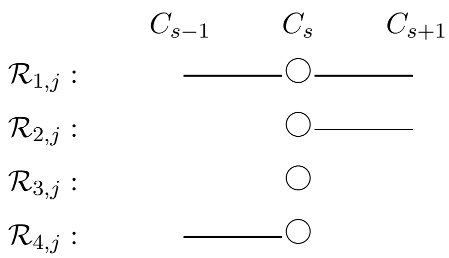
$$\begin{tikzcd}
\mathcal{O}_C(-1)^{\oplus r_2^p} \arrow[rrr, "\bar{\eta}^p"] \arrow[d, hook] & & & \mathcal{O}_C(-1)^{\oplus r_3^{p-1}}[2] \\
\mathcal{H}om_X(\mathcal{O}_C, \bigoplus_j^{r_2^p} \widetilde{\mathcal{R}^p_{2,j}}) \arrow[r, hook] & \bigoplus_j^{r_2^p} \widetilde{\mathcal{R}^p_{2,j}} \arrow[r, "\eta^p"] & \bigoplus_j^{r_3^{p-1}} \widetilde{\mathcal{R}^{p-1}_{3,j}}[2] \arrow[r, two heads] & (\bigoplus_j^{r_3^{p-1}} \widetilde{\mathcal{R}^{p-1}_{3,j}})|_C[2] \arrow[u]
\end{tikzcd}$$

Here the right vertical arrow is the projection to the direct summand. $\square$

Lemma A (cf. Lemma A in [IU05])**.** *Let $\alpha \in D_{Z_0}(S)$, $U = U_s$, and $C = C_s$ be as above. Assume that we can write*

$$\bigoplus_p \mathcal{H}^p(\alpha)|_U = \bigoplus_j^{r_1} \mathcal{R}_{1,j} \oplus \bigoplus_j^{r_2} \mathcal{R}_{2,j} \oplus \bigoplus_j^{r_3} \mathcal{R}_{3,j} \oplus \bigoplus_j^{r_4} \mathcal{R}_{4,j},$$

such that $\mathcal{R}_{k,j} \in \Sigma(\alpha)_U$, and they are of the the following form:



Suppose that either $r_3 \neq 0$ or $r_2 \cdot r_4 \neq 0$ holds, and suppose furthermore that $\operatorname{Supp} \alpha \neq C$*. Then, there is an integer a such that $l(T_{\mathcal{O}_C(a)}(\alpha)) < l(\alpha)$.*

Proof. The proof goes parallel to that of [IU05, Lemma A]. Let us denote by $\widetilde{\mathcal{R}_{1,j}}$ an element in $\Sigma(\alpha)$ such that $\mathcal{R}_{1,j}$ is a direct summand of $\widetilde{\mathcal{R}_{1,j}}|_U$.

When $r_2 = r_4 = 0$, we can see that

$$\chi(\widetilde{\mathcal{R}_{1,j}}, \mathcal{R}_{3,k}) = 0$$

for any j, k. Note that [IU05, Lemma 6.2] is true without any changes in our situation so that we can conclude $r_1 = 0$. This contradicts the asumption that $\operatorname{Supp} \alpha \neq C$. Therefore, by the symmetry, we may safely assume that $r_2 \neq 0$.

In the case $r_2 \cdot r_4 \neq 0$, we can see from Lemma 4.9 that there is an integer a such that

$$\deg_C \mathcal{R}_{2,h} = \deg_C \mathcal{R}_{4,k} = a$$

for all h, k, and that $\deg_C \mathcal{R}_{3,j}$ is a or $a - 1$. Then by Lemma 5.1 (ii) and [IU05, Lemma 4.15], the inequality (22) holds for $\Psi = T_{\mathcal{O}_C(a-1)}$. Hence, we obtain the desired result.

What remains is the case $r_2 \cdot r_3 \neq 0$ and $r_4 = 0$. First, suppose that

$$\deg_{C_s} \mathcal{R}_{3,h} = \max_{k,j} \deg_{C_s} \mathcal{R}_{k,j} (=: b)$$

holds for some h. Then $\deg_{C_s} \mathcal{R}_{1,j} = \deg_{C_s} \mathcal{R}_{2,j} = b$ for all j, and if we put $\Psi = T_{\mathcal{O}_C(b-1)}$, again using Lemma 5.1 (ii) along with [IU05, Lemma 4.15], we obtain the inequality (22).

Next, suppose that $\deg_{C_s} \mathcal{R}_{3,j} = b - 1$ holds for all j. In this case, just apply Lemmas 5.2 and 5.3, which imply the assertion. □

Lemma B (cf. Lemma B in [IU05])**.** *Let $\alpha \in D_{Z_0}(S)$ be a spherical object and fix positive integers s, t with $1 \leq t - s \leq n - 2$. Take a sufficiently small complex analytic neighbourhood $U = U_{s,\dots,t}$ of $C_s \cup \cdots \cup C_t$ and assume that we can write*

$$\bigoplus_p H^p(\alpha)|_U = \bigoplus_j^{r_1} \mathcal{R}_{1,j} \oplus \bigoplus_j^{r_2} \mathcal{R}_{2,j} \oplus \bigoplus_j^{r_3} \mathcal{R}_{3,j} \oplus \bigoplus_j^{r_4} \mathcal{R}_{4,j},$$

where $\mathcal{R}_{k,j} \in \Sigma(\alpha)_U$, and they are of the forms

$$\begin{array}{lcccccccccc}
 & C_{s-1} & & C_s & & C_{s+1} & & C_{t-1} & & C_t & C_{t+1}\,. \\
\mathcal{R}_{1,j}: & \text{———} & & \bigcirc & \text{———} & \bigcirc & \text{———}\cdots\text{———} & \bigcirc & \text{———} & \bigcirc & \text{———} \\
\mathcal{R}_{2,j}: & & & \bigcirc & \text{———} & \bigcirc & \text{———}\cdots\text{———} & \bigcirc & \text{———} & \bigcirc & \text{———} \\
\mathcal{R}_{3,j}: & & & \bigcirc & \text{———} & \bigcirc & \text{———}\cdots\text{———} & \bigcirc & \text{———} & \bigcirc & \\
\mathcal{R}_{4,j}: & \text{———} & & \bigcirc & \text{———} & \bigcirc & \text{———}\cdots\text{———} & \bigcirc & \text{———} & \bigcirc &
\end{array}$$

Suppose that either $r_3 \neq 0$ or $r_2 \cdot r_4 \neq 0$ holds. Then there is

$$\Phi \in \left\langle T_{\mathcal{O}_{C_l}(a)} \;\middle|\; a \in \mathbb{Z}, s \leq l \leq t \right\rangle$$

such that $l(\Phi(\alpha)) < l(\alpha)$.

Proof. The proof goes parallel to that of [IU05, Lemma B]. Note that the straight-forward analogues of [IU05, Lemmas 6.7, 6.8] hold for $H^p(\alpha)|_U$, and use them. $\square$

5.2 The proof of Proposition 4.4

Notice that if we show the existence of an autoequivalence $\Phi \in B_0$ such that $l(\alpha) > l(\Phi(\alpha))$, under the assumption that $l(\alpha) > 1$, then we can prove the statement by induction on $l(\alpha)$. Recall that the proof is already done in [IU05, Proposition 1.7] in the case $\operatorname{Supp}\alpha \neq Z_0$, since in this case α is supported by a chain of projective lines, contained in Z_0. Hence, we may assume $\operatorname{Supp}\alpha = Z_0 = C_1 \cup \cdots \cup C_n$.

For $\mathcal{S} = \mathcal{S}_s(a_s, \dots, a_t) \in \Sigma(\alpha)$, define an integer $s(\mathcal{S})$ by the properties

$$C_{s(\mathcal{S})} = C_s \text{ and } 1 \leq s(\mathcal{S}) \leq n,$$

and an integer $t(\mathcal{S})$ by the properties

$$C_{t(\mathcal{S})} = C_t \text{ and } s(\mathcal{S}) \leq t(\mathcal{S}) \leq s(\mathcal{S}) + n - 2.$$

Here, note that Lemma 4.9 (ii) guarantees that, for $\mathcal{R} \in \Sigma(\alpha)$, there are no elements $\mathcal{S} \in \Sigma(\alpha)$ such that $C_{t(\mathcal{S})} = C_{s(\mathcal{R})-1}$ or $C_{t(\mathcal{R})} = C_{s(\mathcal{S})-1}$. Thus, we have

$$l_{s(\mathcal{R})-1}(\alpha) < l_{s(\mathcal{R})}(\alpha) \text{ and } l_{t(\mathcal{R})}(\alpha) > l_{t(\mathcal{R})+1}(\alpha).$$

Therefore, we can find integers s_0 and t_0 such that

$$l_{s_0-1}(\alpha) < l_{s_0}(\alpha) = l_{s_0+1}(\alpha) = \cdots = l_{t_0}(\alpha) > l_{t_0+1}(\alpha).$$

Then we are in the situation of Lemma A (if $s_0 = t_0$) or Lemma B (if $s_0 < t_0$). So the proof is done.

Remark 5.4. Take an arbitrary element $\mathcal{R} \in \Sigma(\alpha)$. Then, in the proof above, we can find s_0, t_0 such that $s(\mathcal{R}) \leq s_0 \leq t_0 \leq t(\mathcal{R})$. Thus, Lemma A or B provides

$$\Phi \in \left\langle T_{\mathcal{O}_{C_l}(a)} \;\middle|\; a \in \mathbb{Z}, s(\mathcal{R}) \leq l \leq t(\mathcal{R}) \right\rangle$$

such that $l(\alpha) > l(\Phi(\alpha))$. We shall use this remark in §6.

6 The proof of Proposition 4.2

6.1 Facts needed for the proof of Proposition 4.2

Let Φ be an autoequivalence of $D_{Z_0}(S)$ that preserves the cohomology class of a point. Put $\alpha = \Phi(\mathcal{O}_{C_1})$ and $\beta = \Phi(\mathcal{O}_{C_1}(-1))$. Applying Proposition 4.4 and the shift functor [1], we may assume that

$$\alpha = \mathcal{O}_{C_b}(a)$$

for some $a, b \in \mathbb{Z}$ with $1 \le b \le n$. To prove Proposition 4.2, it suffices to show the following;

Claim 6.1. *Suppose that $l(\beta) > 1$. There is an autoequivalence $\Psi \in B_0$ such that $l(\Psi(\alpha)) = 1$ and $l(\beta) > l(\Psi(\beta))$.*

In fact, Proposition 4.2 easily follows from this:

Proof of Proposition 4.2. By Claim 6.1, we can reduce the problem to the case $l(\alpha) = l(\beta) = 1$. In this case, the supports of α and β must be the same by Condition 6.5 below. Therefore, we get the conclusion from the A_1-case [IU05, Proposition 1.6], and we can complete the proof of Proposition 4.2 by induction on $l(\beta)$. □

The most of the proof of Claim 6.1 goes parallel to that of [IU05, Claim 7.1].

Fact 6.2 (cf. Condition 7.2 in [IU05]). *We may assume*

$$\max\{\deg_{C_b} \mathcal{R} \mid \mathcal{R} \in \Sigma(\beta)_{U_b}, \operatorname{Supp} \mathcal{R} \supset C_b\} = 0.$$

Especially, $\deg_{C_b} \mathcal{R} = 0$ *or* -1 *for all* $\mathcal{R} \in \Sigma(\beta)_{U_b}$ *with* $\operatorname{Supp} \mathcal{R} \supset C_b$ *by Lemma 4.9 (iii-1).*

The relations between $\mathcal{O}_{C_1}$ and $\mathcal{O}_{C_1}(-1)$ impose conditions on a and β. From the spectral sequence

$$E_2^{p,q} = \operatorname{Ext}_S^p(H^{-q}(\beta), \mathcal{O}_{C_b}(a)) \Longrightarrow \operatorname{Hom}^{p+q}_{D(S)}(\beta, \alpha) = \begin{cases} \mathbb{C}^2 & p+q=0 \\ 0 & p+q \neq 0, \end{cases} \tag{23}$$

we obtain

Fact 6.3 (cf. Condition 7.3 in [IU05]). $E_2^{1,q} = 0$ *for* $q \neq -1$

and

Fact 6.4 (cf. Condition 7.4 in [IU05]). $d_2^{0,-1} : E_2^{0,-1} \to E_2^{2,-2}$ *is injective,* $d_2^{0,0} : E_2^{0,0} \to E_2^{2,-1}$ *is surjective, and* $d_2^{0,q} : E_2^{0,q} \to E_2^{2,q+1}$ *are isomorphisms for all* $q \neq 0, -1$.

In addition to Facts 6.3 and 6.4, (23) implies

$$\dim \operatorname{Coker} d_2^{0,-1} + \dim \operatorname{Ker} d_2^{0,0} + \dim E_2^{1,-1} = 2. \tag{24}$$

To show the following, we need the assumption that Φ preserves the cohomology class $\operatorname{ch}(\mathcal{O}_x)$ in $H^4(S, \mathbb{Q})$.[5]

[5]If we drop this assumption, we can just conclude $c_1(\beta) = [C_b] + k[Z_0]$ for some $k \in \mathbb{Z}$ by a similar proof to that of Condition 7.5 in [IU05].

Fact 6.5 (cf. Condition 7.5 in [IU05])**.** *The equality $c_1(\beta) = [C_b]$ holds in the cohomology group $H^2(S, \mathbb{Q})$.*

Proof. By the choice of Φ, we have

$$0 = c_1(\mathcal{O}_x) = c_1(\Phi(\mathcal{O}_x)) = c_1(\alpha) - c_1(\beta),$$

which gives the assertion. $\square$

Claim 6.6 (cf. Claim 7.6 in [IU05])**.** *We have $a \geq -1$.*

Proof. The proof is similar to that of [IU05, Claim 7.6]. $\square$

Claim 6.7 (cf. Claim 7.7 in [IU05])**.** *Fix $q \neq 0$. If $E_2^{2,-q-1} = 0$ in (23), then we have $\deg_{C_b} \mathcal{R} > a$ for all direct summands $\mathcal{R} \in \Sigma(\beta)_{U_b}$ of $H^q(\beta)|_{U_b}$ with $\operatorname{Supp} \mathcal{R} \supset C_b$. If, in addition, we suppose that $a \geq 0$, then we get $C_b \not\subset \operatorname{Supp} H^q(\beta)$.*

Proof. The proof of Claim 6.7 is similar to that of [IU05, Claim 7.7]. $\square$

Remark 6.8. Below we use the notation $t(\mathcal{R})$ and $s(\mathcal{R})$ for $\mathcal{R} \in \Sigma(\beta)$ defined in §5.2, and recall that b satisfies $1 \leq b \leq n$ by the definition.

If there is an element $\mathcal{R} \in \Sigma(\beta)$ with either

$$t(\mathcal{R}) - n + 1 < b < s(\mathcal{R}) - 1 \text{ or } t(\mathcal{R}) + 1 < b,$$

then we can find $\Psi \in \left\langle T_{\mathcal{O}_{C_l}(a)} \mid a \in \mathbb{Z}, C_l \subset \operatorname{Supp} \mathcal{R} \right\rangle$ such that $\Psi(\alpha) \cong \alpha$ and $l(\Psi(\beta)) < l(\beta)$ by Remark 5.4. Therefore, we may assume that

$$s(\mathcal{R}) - 1 \leq b \leq t(\mathcal{R}) + 1 \text{ or } b + n \leq t(\mathcal{R}) + 1$$

for all $\mathcal{R} \in \Sigma(\beta)$.

Now we divide the proof into cases as in [IU05, Division into cases on page 426]. We have only to consider the three cases:

Division into Cases. (i) $C_b \subset \operatorname{Supp} \mathcal{R}$ for all $\mathcal{R} \in \Sigma(\beta)_{U_b}$,

(ii) there is an $\mathcal{R} \in \Sigma(\beta)_{U_b}$ with $\operatorname{Supp} \mathcal{R} \cap C_b = C_{b+1} \cap C_b$, but there is no $\mathcal{R}' \in \Sigma(\beta)_{U_b}$ with $\operatorname{Supp} \mathcal{R}' \cap C_b = C_{b-1} \cap C_b$,

(iii) there are $\mathcal{R}, \mathcal{R}' \in \Sigma(\beta)_{U_b}$ with $\operatorname{Supp} \mathcal{R} \cap C_b = C_{b+1} \cap C_b$ and $\operatorname{Supp} \mathcal{R}' \cap C_b = C_{b-1} \cap C_b$.

6.2 Case (i)

In this case, we can find Ψ in Claim 6.1 in a similar way to that of [IU05, §7.3. Case (i)], after some obvious changes; for instance [IU05, Claim 7.8] should be replaced by

Claim 6.9.

$$\mathcal{O}_{\cdots \cup C_b}(*, -1)|_{U_b}, \mathcal{O}_{C_b \cup \cdots}(-1, *)|_{U_b}, \mathcal{O}_{\cdots \cup C_b \cup \cdots}(*, -1, *)|_{U_b} \notin \Sigma(\beta)_{U_b}.$$

6.3 Case (ii)

The existence of $\mathcal{R} \in \Sigma(\beta)_{U_b}$ with $\operatorname{Supp}\mathcal{R} \cap C_b = C_b \cap C_{b+1}$ and Lemma 4.9 (ii) imply the non-existence of $\mathcal{S} \in \Sigma(\beta)_{U_b}$ with $\operatorname{Supp}\mathcal{S} \cap C_{b+1} = C_b \cap C_{b+1}$. Note that $n > 2$ in this case by Lemma 4.8 (i). Furthermore, we have

$$\Sigma(\beta)_{U_b} \subset \left\{ \mathcal{O}_{C_b \cup \cdots}(a', *)|_{U_b}, \mathcal{O}_{C_{b+1}}(*)|_{U_b}, \mathcal{O}_{\cdots \cup C_b \cup \cdots}(*, a', *)|_{U_b} \;\middle|\; a' = -1, 0 \right\}.$$

Then we can find Ψ in Claim 6.1 in a similar way to that of [IU05, Case (ii)].

6.4 Case (iii)

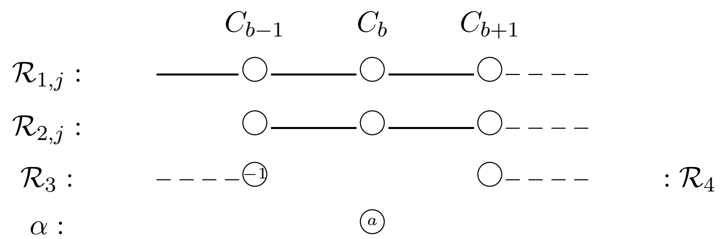
Fact 6.3 implies that $\mathcal{R}$ and $\mathcal{R}'$ above must be in $H^1(\beta)$. Moreover, they are unique in a decomposition of $H^1(\beta)$, by virtue of the inequality $\dim E_2^{1,-1} \leq 2$ from (24).

We can exclude the case $n = 2$ as follows. Suppose that $n = 2$. Recall that $\mathcal{R}$ is the unique element in $\Sigma(\beta)$ such that $C_{s(\mathcal{R})} \neq C_b$, and $\mathcal{R}'$ is the unique element in $\Sigma(\beta)$ such that $C_{t(\mathcal{R}')} \neq C_b$. It follows from Lemma 4.8 (i) that every element $\mathcal{S} \in \Sigma(\beta)$ satisfies that $C_{s(\mathcal{S})} = C_{t(\mathcal{S})}$. Therefore $\beta[-1]$ is an indecomposable sheaf $\mathcal{S} (= \mathcal{R} = \mathcal{R}')$ satisfying $C_{s(\mathcal{S})} = C_{t(\mathcal{S})} \neq C_b$. In this case, Fact 6.5 cannot be satisfied.

Now, Lemma 4.9 allows us to write

$$\bigoplus_p H^p(\beta)|_{U_{b-1,b}} = \bigoplus_j^{r_1} \mathcal{R}_{1,j} \oplus \bigoplus_j^{r_2} \mathcal{R}_{2,j} \oplus \mathcal{R}_3 \oplus \mathcal{R}_4,$$

where $\mathcal{R}_{k,j}$'s, $\mathcal{R}_3$ and $\mathcal{R}_4$ are sheaves in $\Sigma(\beta)_{U_{b-1,b}}$ of the following forms:



Here, we assume that $\deg_{C_{b-1}} \mathcal{R}_3 = -1$ for simplicity.

Claim 6.10. *We have $a = -1$.*

Proof. A similar proof to that of [IU05, Claim 7.11] works. □

The inequality $\dim E_2^{1,-1} \leq 2$ from (24) also implies that

$$\operatorname{Ext}^1_S(\mathcal{R}_{k,j}, \mathcal{O}_{C_b}(-1)) = 0$$

for $k = 1, 2$ and for all j. In particular, we get

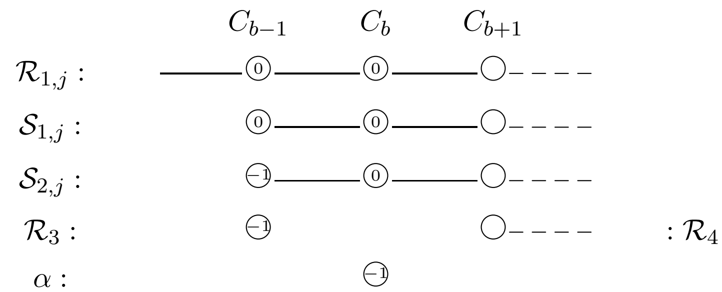
$$\deg_{C_b} \mathcal{R}_{1,j} = \deg_{C_b} \mathcal{R}_{2,j} = 0.$$

Recall that there is a unique sheaf $\mathcal{R} \in \Sigma(\beta)$ satisfying $C_{t(\mathcal{R})} = C_{b-1}$, and $\mathcal{R}|_{U_{b-1,b}}$ contains $\mathcal{R}_3$ as a direct summand. Now, we give a proof for Case (iii) by induction on $l(\mathcal{R})$.

First, suppose $l(\mathcal{R}) = 1$. In this case $\mathcal{R} = \mathcal{R}_3$, and we write

$$\bigoplus_j^{r_2} \mathcal{R}_{2,j} = \bigoplus_j^{s_1} \mathcal{S}_{1,j} \oplus \bigoplus_j^{s_2} \mathcal{S}_{2,j},$$

where $\mathcal{S}_{k,j}$'s are sheaves in $\Sigma(\beta)_{U_{b-1,b}}$ of the following forms.



Note that we may assume $r_1 \neq 0$ since otherwise $\operatorname{Supp} \beta$ is strictly smaller than Z_0, in which case [IU05, Claim 7.1] gives the result. Because of the existence of $\mathcal{R}_3$, we have $r_2 \neq 0$ by [IU05, Lemma 6.2], which continues to hold in our setting. Hence, $s_1 \neq s_2$ by Lemma 5.3. Define

$$\Psi_0 = \begin{cases} T_{\mathcal{O}_{C_{b-1} \cup C_b}(-1,-1)} & \text{if } s_1 < s_2, \\ T'_{\mathcal{O}_{C_{b-1} \cup C_b}} & \text{if } s_2 < s_1. \end{cases}$$

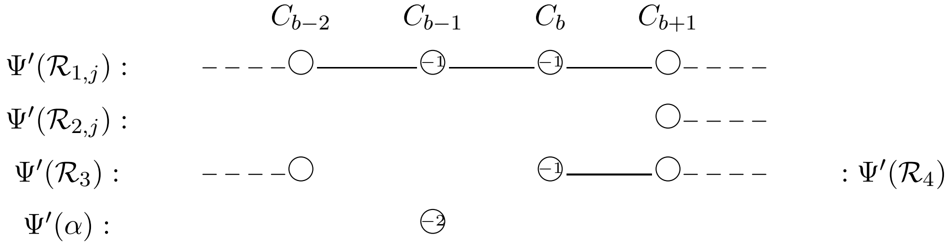
Then $(\Psi_0(\alpha), \Psi_0(\beta))$ fits in Case (ii) and $\Psi_0(\beta)$ satisfies $l(\Psi_0(\beta)) \leq l(\beta)$. Since we have proved Case (ii), this finishes the case $l(\mathcal{R}) = 1$.

Next, suppose $l(\mathcal{R}) > 1$. In this case, Lemma 4.9 (iii-4) implies

$$\deg_{C_{b-1}} \mathcal{R}_{2,j} = -1.$$

Set $\Psi' = T_{\mathcal{O}_{C_b}(-1)} \circ T_{\mathcal{O}_{C_{b-1}}(-2)}$. Then we have

$$\Psi'(\alpha) \cong \mathcal{O}_{C_{b-1}}(-2) \text{ and } l(\Psi'(\beta)) \leq l(\beta).$$



First, let us consider the case $C_{s(\mathcal{R})} = C_{b+1}$, equivalently $\mathcal{R} = \mathcal{R}'$. Take an element $\widetilde{\mathcal{R}_{2,j}} \in \Sigma(\beta)$ such that $\widetilde{\mathcal{R}_{2,j}}|_{U_{b-1,b}}$ contains $\mathcal{R}_{2,j}$ as a direct summand. Then $\Psi'(\widetilde{\mathcal{R}_{2,j}})$ satisfies

$$t(\Psi'(\widetilde{\mathcal{R}_{2,j}})) - n + 1 < b - 1 < s(\Psi'(\widetilde{\mathcal{R}_{2,j}})) - 1$$

or

$$t(\Psi'(\widetilde{\mathcal{R}_{2,j}})) + 1 < b - 1.$$

Hence, we can find Ψ as in Claim 6.1 by Remark 6.8. Therefore, we may assume that $C_{s(\mathcal{R})} \neq C_{b+1}$. Note that $\Psi'(H^q(\beta))$ is a sheaf for every $q \in \mathbb{Z}$. Using the spectral sequence

$$E_2^{p,q} = H^p(\Psi'(H^q(\beta))) \Longrightarrow E^{p+q} = H^{p+q}(\Psi'(\beta)),$$

we see $\Psi'(H^q(\beta)) = H^q(\Psi'(\beta))$, and then $(\Psi'(\alpha), \Psi'(\beta))$ fits in Case (iii). Furthermore, the assumption $C_{s(\mathcal{R})} \neq C_{b+1}$ implies that $l(\Psi'(\mathcal{R})) < l(\mathcal{R})$. Hence, we can conclude by induction on $l(\mathcal{R})$.

□

7 Example

If $\lambda_S \leq 4$, we have $H_S = \{\mathrm{id}_S\}$ and $\operatorname{Im}\Theta = \mathrm{SL}(2,\mathbb{Z})$ as in Remark 3.12 (ii). For a general elliptic surface S, however, it is not easy to describe the group H_S concretely. In this section, we give an example of elliptic surfaces with $\lambda_S \geq 5$ where we are able to determine H_S, and hence $\operatorname{Im}\Theta$, more concretely.

Let us consider a rational elliptic surface $\pi\colon J \to \mathbb{P}^1$ with a section, and assume that π has four singular fibers of types I_7, I_2, II and I_1. Such a surface exists by Persson's list [Pe90]. Take a point $s \in \mathbb{P}^1$ over which the fiber of π is not of type II. Apply a logarithmic transformation along the point s to obtain a rational elliptic surface S whose Jacobian surface is J, and S has a multiple fiber of the multiplicity m over the point s. Suppose that $m > 2$. Then as in [Ue11, Example 2.6], we can show that $H_S = \{\pm 1\}$ (we leave it to the reader to check this). Therefore, Theorem 1.3 assures that there is a short exact sequence:

$$1 \to \langle B, \otimes\mathcal{O}_S(D) \mid D\cdot F = 0\rangle \rtimes \operatorname{Aut} S \times \mathbb{Z}[2] \to \operatorname{Auteq} D(S)$$
$$\xrightarrow{\Theta} \Big\{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \Gamma_0(m) \;\Big|\; b \equiv \pm 1 \pmod{m} \Big\} \to 1.$$

In this case, $\operatorname{Aut} S = \operatorname{Aut}_{\mathbb{P}^1} S$ is just the semi-direct product of the Mordell-Weil group of S and the subgroup of automorphisms preserving the zero

section (cf. [FM94, Theorem 1.3.14]). Hence, it can be calculated by using [OS90].

In the upcoming paper [Ue], we consider the autoequivalence group and Fourier–Muaki partners of elliptic ruled surfaces.

References

[At57] M. Atiyah, Vector bundles over an elliptic curve, Proc. London Math., Soc. 5 (1955), 407–434.

[BBDG] L. Bodnarchuk, I. Burban, Y. Drozd, G.-M. Greuel, Vector bundles and torsion free sheaves on degenerations of elliptic curves. Global aspects of complex geometry, 83–128, Springer, Berlin, 2006.

[BHPV] Barth, Wolf P.; Hulek, Klaus; Peters, Chris A. M.; Van de Ven, Antonius, Compact complex surfaces. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 4. Springer-Verlag, Berlin, 2004. xii+436 pp.

[BO95] A.I. Bondal, D.O. Orlov, Semiorthogonal decomposition for algebraic varieties, alg-geom 9712029.

[Br98] T. Bridgeland, Fourier–Mukai transforms for elliptic surfaces. J. Reine Angew. Math. 498 (1998), 115–133.

[Br99] T. Bridgeland, Equivalences of triangulated categories and Fourier–Mukai transforms, Bull. London Math. Soc. 31 (1999), 25–34.

[BM98] T. Bridgeland, A. Maciocia, Fourier-Mukai transforms for quotient varieties. math.AG/9811101.

[BM01] T. Bridgeland, A. Maciocia, Complex surfaces with equivalent derived categories. Math. Z. 236 (2001), 677–697.

[BM02] T. Bridgeland, A. Maciocia, Fourier-Mukai transforms for K3 and elliptic fibrations. J. Algebraic Geom. 11 (2002), no. 4, 629–657.

[BV03] A. Bondal, M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry. Mosc. Math. J. 3 (2003), no. 1, 1–36, 258.

[Fr95] R. Friedman, Vector bundles and SO(3)-invariants for elliptic surfaces, J. Amer. Math. Soc. 8 (1995), 29–139.

[FM94] R. Friedman, J. W. Morgan, Smooth Four-Manifolds and Complex Surfaces, Springer–Verlag Berlin Heidelberg 1994.

[Ha66] R. Hartshorne, Residues and duality. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20 Springer-Verlag, Berlin-New York 1966 vii+423 pp.

[Ha77] R. Hartshorne, Algebraic Geometry, Springer–Verlag, Berlin Heidelberg New York, 1977.

[HLS09] D. Hernández Ruipérez, A.C. López Martín, F. Sancho de Salas, Relative integral functors for singular fibrations and singular partners. J. Eur. Math. Soc. (JEMS) 11 (2009), no. 3, 597–625.

[Hu06] D. Huybrechts, Fourier-Mukai transforms in algebraic geometry. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006. viii+307 pp.

[IU05] A. Ishii, H. Uehara, Autoequivalences of derived categories on the minimal resolutions of A_n-singularities on surfaces. J. Differential Geom. 71 (2005), no. 3, 385–435.

[IUU10] A. Ishii, K. Ueda, H. Uehara, Stability conditions on A_n-singularities. J. Differential Geom. 84 (2010), no. 1, 87–126.

[Ka02] Y. Kawamata, D-equivalence and K-equivalence. J. Differential Geom. 61 (2002), 147–171.

[KO95] S. A. Kuleshov, D. Orlov, Exceptional sheaves on Del Pezzo surfaces. Russian Acad. Sci. Izv. Math. 44 (1995), no. 3, 479–513

[LST13] A.C. López Martín, D. Sánchez Gómez, C. Tejero Prieto, Relative Fourier–Mukai functors for Weierstraß fibrations, abelian schemes and fano fibrations, Math. Proc. Cambridge Philos. Soc. 155 (2013) 129–153.

[OS90] K. Oguiso, T. Shioda, The Mordell–Weil lattice of a rational elliptic surface, Comment. Math. Univ. St. Pauli 40 (1991) 83–99.

[Or97] D. Orlov, Equivalences of derived categories and K3 surfaces. Algebraic geometry, 7. J. Math. Sci. (New York) 84 (1997), 1361–1381.

[Or02] D. Orlov, Derived categories of coherent sheaves on abelian varieties and equivalences between them, Izv. Math. 66 (2002) 569–594.

[Pe90] U. Persson, Configurations of Kodaira fibers on rational elliptic surfaces, Math. Z. 205 (1) (1990) 1–47.

[ST01] P. Seidel, R. Thomas, Braid group actions on derived categories of coherent sheaves, Duke Math. J. 108 (2001), 37–108.

[To03] Y. Toda, Fourier-Mukai transforms and canonical divisors. Compos. Math. 142 (2006), no. 4, 962–982.

[Ue04] H. Uehara, An example of Fourier-Mukai partners of minimal elliptic surfaces. Math. Res. Lett. 11 (2004), no. 2-3, 371–375.

[Ue11] H. Uehara, A counterexample of the birational Torelli problem via Fourier-Mukai transforms. J. Algebraic Geom. 21 (2012), no. 1, 77–96.

[Ue] H. Uehara, Derived categories of elliptic ruled surfaces (preprint).

Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1 Minamiohsawa, Hachioji-shi, Tokyo, 192-0397, Japan

e-mail address : hokuto@tmu.ac.jp